Towards a Gauge Theory in Finite Projective Physics

Master's Thesis in Physics

Presented by

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Abstract

One of the most challenging issues of contemporary theoretical physics is the unification of Einstein's General Relativity with Quantum Field Theory, most notably known as the conceptual framework of the Standard Model of Particle Physics. In this new attempt of Finite Projective Physics, an event-based ontology is employed where sequences of events are the main point of interest. This leads to an embedding into a projective space $\mathbb{F}_q\mathrm{P}^n$ over a finite field $\mathbb{F}_q$ equipped with a (bi-)quadric field such that predecessors and successors of an event are elements of the quadric at the event and lie on a straight line through the centre of this quadric by the law of inertia. Relativistic Newtonian mechanics can be described naturally by an additional translation of a flat and force-free quadric field in the physical affine subspace proportional to the force acting on the body.

In this thesis, we at first review and explain necessary mathematical concepts like finite fields and projective spaces and add additional explanatory theorems which may not be found in text books on these topics. Furthermore, we provide an overview over the description of relativistic Newtonian mechanics in the framework of Finite Projective Physics and explicitly construct the trajectory of a massive body under the influence of an arbitrary force field. Using the Lorentz force of a constant electromagnetic field, we show that the relativistic motion of a massive and charged body in this electromagnetic field in dimension $n = 4$ can be recovered using this new ansatz, but deviations from the analytical solution arise in the ultra relativistic regime on longer time scales. We show that the use of non-constant electromagnetic fields in our description leads in particular to the requirement of a constant magnitude of the electric field and a constant Poynting vector along the trajectory. In the next chapter, we describe various ideas of a geometrical implementation of gauge theories using in particular symmetry groups of the intersection of two quadrics and thereof derived geometrical structures like the hyperplane of intersection, and show that in two dimensions in the case of quadrics translated with respect to each other with respect to some hyperplane at infinity the symmetry group of the intersection is isomorphic to the symmetric group S_4 on the four elements of intersection. In higher dimensions, we show that not every permutation of points of this intersection is possible. Afterwards, we describe an idea of how to implement charges by means of matrices which are invariant under conjugacy with certain transformations. Lastly, we discuss how an introduction of gravity could help to eliminate the problems which arise in the case of non-constant electromagnetic fields, and explain possible benefits of changing the theory from a theory of translations to a theory of lines.

Zusammenfassung

Eine der größten Herausforderungen der modernen theoretischen Physik ist die Vereinigung von Einsteins Allgemeiner Relativitätstheorie mit der Quantenfeldtheorie, die insbesondere als Grundlage der Beschreibung des Standardmodells der Teilchenphysik fungiert. In einem neuen Ansatz, Finite Projective Phyics genannt, werden durch eine ereignisbasierte Ontologie Sequenzen von Ereignissen Gegenstand der Physik. Diese lassen sich in einen projektiven Raum $\mathbb{F}_q\mathrm{P}^n$ über einem endlichen Körper $\mathbb{F}_q$ einbetten, der mit einem (Bi-)Quadrikfeld ausgestattet ist. Vorgänger und Nachfolger eines Ereignisses werden durch das Trägheitsprinzips als Punkte der Quadrik am Ereignis selbst bestimmt, die auf der Geraden durch das Zentrum der Quadrik liegt. Mithilfe dieser Idee lässt sich die relativistische Newtonsche Mechanik geometrisch beschreiben, indem man ein flaches und kräftefreies Quadrikfeld annimmt und dieses an jedem Ort proportional zur wirkenden Kraft im physikalischen affinen Unterraum verschiebt.

In dieser Arbeit werden zunächst für das Verständnis notwendige mathematische Konzepte wie endliche Körper oder projektive Räume wiederholt und erklärt, wobei eigene, erklärende Sätze hinzugefügt wurden. Anschließend wird ein Überblick über die Beschreibung der relativistischen Newtonschen Mechanik im Rahmen vdieser Theorie gegeben und die Weltlinie eines massiven Körpers, auf den ein beliebiges Kraftfeld wirkt, explizit konstruiert. Mithilfe der Lorentzkraft eines konstanten elektromagnetischen Feldes in Dimension $n = 4$ kann anschließend anhand von Beispielen gezeigt werden, dass die Bewegung eines massiven geladenen Körpers in diesem Feld über diesen Ansatz ermittelt werden kann, wobei Abweichungen zur analytischen Lösung vor allem im ultrarelativistischen Bereich auf längeren Zeitskalen entstehen. Es wird gezeigt, dass nichtkonstante elektromagnetische Felder in dieser Beschreibung insbesondere zur Forderung der Konstanz des Poynting-Vektors und der Stärke des elektrischen Feldes entlang der Weltlinie führen. Im nachfolgenden Kapitel werden verschiedene Ideen beschrieben, wie Eichtheorien geometrisch in diese Theorie eingebaut werden könnten. Von besonderem Interesse sind dabei die Symmetriegruppen von Schnitten von Quadriken und von davon abgeleiteten geometrischen Strukturen wie die Schnitthyperebene. Hierbei wird gezeigt, dass in zwei Dimensionen im Falle von gegeneinander bezogen auf eine Hyperebene im Unendlichen verschobenen Quadriken die Symmetriegruppe dieses Schnitts isomorph zur symmetrischen Gruppe S_4 der vier Elemente dieses Schnitts ist. In höheren Dimensionen lässt sich dann zeigen, dass nicht alle Permutationen von Elementen des Schnitts möglich sind. Anschließend wird eine Idee zur Implementierung von Ladungen via unter Konjugation mit bestimmten Transformationen invarianter Matrizen diskutiert. Zuletzt wird argumentiert, wie die Einführung von Gravitation die Probleme im Falle von nichtkonstanten elektromagnetischen Feldern verhindern könnte und dass es Vorteile bringen könnte, von einer Theorie der Verschiebungen zu einer Theorie der Geraden zu wechseln.

Contents

1 Introduction

In the last century, in particular two theories emerged and rose to change the view on our universe with mathematical beauty: General Relativity and the Standard Model of Particle Physics. Both are experimentally tested to very high precision and provide in their own right a description of physics on big and on small scales, respectively. Einstein's General Relativity geometrically explains the phenomenon of gravity by the interaction of matter with the spacetime geometry which then has implications on the motion of the matter. On the other hand, the Standard Model of Particle Physics explains in a unified matter how the three fundamental forces, the electromagnetic, weak and strong force, act on and build up matter, and how these are carried. However, General Relativity is a classical theory whereas the Standard Model and with it matter is formulated within the framework of Quantum Field Theory.

For years, researches have tried to find a theory which unifies these two in order to describe all scales of the universe within one framework. Prominent examples include String Theories and Loop Quantum Gravity. The goal is also to predict phenomena which may have occurred at the beginning of our universe in order to better understand its evolution. However, none of these attempts is widely accepted or yielded experimentally testable results which proved to be true.

In this new attempt of Finite Projective Physics by Klaus Mecke, the description of the universe is event-based, instead of the usual matter-based ontology. These events are to be understood as events of creation and only their sequences of succession have physical meaning. They can then be embedded into a projective space over a finite field with a (bi-)quadric field imposed onto it which will yield possible successors and predecessors for every point in this space. Within this framework, special relativistic mechanics can be described purely geometrically without the use of differential equations, and additional so-called points at infinity provide extra degrees of freedom for the description of physical theories.

In this thesis, we will at first provide a review of important mathematical concepts which are used in Finite Projective Physics. Afterwards, we will introduce our definition of a spacetime and model special relativistic mechanics on it. Within this framework, we will present results regarding the behaviour of a charged body in an electromagnetic field using the Lorentz force for constant and non-constant electromagnetic fields. We will then discuss multiple ideas of how to implement gauge transformations within Finite Projective Physics using symmetry groups of the intersection of quadrics and other techniques. This will be followed by a short overview and discussion of open questions and problems and new ideas which may be the starting point of future research.

2 Mathematical Preliminaries

Before we can get to the more physical topics of this thesis, we need to recall and introduce some mathematical concepts and notions upon which our theory and models will be based. We also present some important theorems and results and will give explanatory examples.

Hereinafter, we will discuss and explain the concepts of groups, fields, in particular finite fields, projective and affine spaces with different approaches, quadratic forms and (projective) quadrics.

We will assume a basic exposure to topics of (linear) algebra, such as vector spaces or matrices, but nevertheless recall and discuss some important definitions and theorems. As general references for these definitions and proofs of the theorems we refer to, e. g. [9], [7], or [2] and [3] for the study of projective and affine geometries.

2.1 Groups

The mathematical concept of a *group* is omnipresent in both mathematics and physics. It is for example used in the description of the addition of integers or in the description of symmetries of physical systems. The theory behind it, also known as *group theory*, is broadly developed and will come in handy in the analysis of the constructions of our physical models and theory which will be discussed later.

A group as we would like to use it is defined as follows.

Definition 2.1.1.

1. A **group** $(G, \cdot)$ is a set G equipped with a binary operation $\cdot : G \times G \to G, (x, y) \mapsto x \cdot y$, called the **multiplication** of G, such that the following three axioms are satisfied:

   (G1) **Associativity:** $\forall g, h, k \in G\colon (g \cdot h) \cdot k = g \cdot (h \cdot k)$.

   (G2) **Existence of a neutral element:** $\exists\, e \in G\colon \forall g \in G\colon e \cdot g = g \cdot e = g$.
   e is called **neutral element** of G, often denoted e_G.

   (G3) **Existence of inverse elements:** $\forall g \in G \;\exists\, h \in G\colon g \cdot h = h \cdot g = e$.
   h is then called the **inverse** of g, denoted $h = g^{-1}$.

2. A group G is called **abelian** if $\forall g, h \in G\colon g \cdot h = h \cdot g$. The multiplication $\cdot$ is then said to be **commutative**.

3. The number of elements of G is called **order** of the group $(G, \cdot)$, denoted $|G|$. If $|G| = n < \infty$, $(G, \cdot)$ is called **finite group of order** n.

Remark 2.1.2.

1. There are several other notions of a set equipped with a binary operation which are obtained by omitting some of the axioms of a group. For example, if the multiplication only satisfies (G1), we call this structure **semigroup**. If it satisfies (G1) and (G2), it is called **monoid**.

2. It is common practice to drop the explicit mention of the multiplication when talking about a group. Thus, a group $(G, \cdot)$ is often referred to as G. Furthermore, the multiplication of two elements $g, h \in G$ is usually just denoted gh instead of $g \cdot h$. Furthermore, the notation $g^n := \underbrace{g \cdots g}_{n\times}$ is used.

3. Associativity assures that the order of the multiplication, which is a binary operation, of three or more elements is irrelevant. Hence, we often let go of brackets and only use them for better readability.

4. It can be proven by a straightforward calculation that the neutral element $e \in G$ of a group G as well as the inverse of an element $g \in G$ are unique, respectively. Thus, we can talk about *the* neutral element of G or *the* inverse element of an element $g \in G$. Therefore, the notation g^{-1} for the inverse element of $g \in G$ has its Raison d'être and is widely used.

5. In an abelian group G the multiplication is usually denoted $+$ to show its commutative character. The neutral element of an abelian group is then called zero element and denoted 0 and the inverse of an element $g \in G$ is denoted $-g$ instead of g^{-1}.

6. As a basic result of group theory we get in a group G that $e_G^{-1} = e_G$ and $\forall g \in G\colon\ (g^{-1})^{-1} = g$ by exchanging g with g^{-1} in the definition of the inverse element.

To get a better grasp of what a group really is and where this concept appears in mathematics and physics we present a few common examples.

Example 2.1.3.

1. The minimal example of a group is a set with just one element $G = \{e\}$ which serves as the neutral element. This is called the **trivial group**. It is also obviously abelian and of finite order.

2. For two groups $(G, \cdot_G), (H, \cdot_H)$, the **direct product** $(G \times H, \cdot_{G\times H})$ is again a group where $G \times H := \{(g, h) \mid g \in G, h \in H\}$ and the multiplication is given by $(g_1, h_1) \cdot_{G\times H} (g_2, h_2) := (g_1 \cdot_G g_2, h_1 \cdot h_2)$ for $g_1, g_2 \in G, h_1, h_2 \in H$ with neutral element $e_{G\times H} := (e_G, e_H)$.

3. The integers $(\mathbb{Z}, +)$ with addition as group multiplication form an abelian group. This group is of infinite order.

Notice that the subset of natural numbers including 0, $\mathbb{N}_0 \subsetneq \mathbb{Z}$, does not prove to be a group since it does not satisfy the axiom of the existence of inverse elements, e. g. $-1 \notin \mathbb{N}_0$, even though it has a neutral element.

4. Let $n \in \mathbb{N}$ be some natural number. Integers modulo n, $\mathbb{Z}/n\mathbb{Z} = \{[0], [1], \ldots, [n-1]\}$, equipped with addition modulo n forms an abelian group. For n prime, these will later be proven to be so-called finite fields.
The order of $\mathbb{Z}/n\mathbb{Z}$ is given by $|(\mathbb{Z}/n\mathbb{Z})| = n$, so it is a finite group of order n.

5. The set of invertible $(n \times n)$-matrices over the real numbers $\mathbb{R}$, denoted $\mathrm{GL}(n, \mathbb{R})$, with the usual matrix multiplication yields a group of infinite order. It is non-abelian in the case of $n > 1$.
In the case of $n = 1$ it is just the group of units of the real numbers, i. e. $\mathrm{GL}(1, \mathbb{R}) = \mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ with the usual multiplication of real numbers.

6. The symmetries of d-dimensional Minkowski space $\mathbb{R}^{1,d}$ leaving the Minkowski metric $\eta = \mathrm{diag}(-1, 1, \ldots, 1)$ invariant form a (non-abelian) group, the so-called Lorentz group $\mathrm{O}(1, d)$.
Here, the elements can be thought of as $((d+1)\times(d+1))$-matrices $L \in \mathrm{O}(1, d)$ which fulfil the equation $L^T \eta L = \eta$ where the superscript $\cdot^T$ denotes transposition. It is also a so-called *Lie group* which is a smooth manifold where both multiplication and inversion are smooth maps in the sense of differential geometry.

When studying algebraic concepts, it often proves to be useful to not only study the objects of interest but also to introduce maps between two entities of this concept which preserve their structure. These are known as *(homo)morphism* in the underlying category of this concept. The advantage is that we do not break up any of the established structures by this map and can use these maps to study properties of very abstract objects by mapping those to more suitable and more easy to handle ones.

In the case of groups, these maps are called *group homomorphisms* and are defined as follows.

Definition 2.1.4. Let $(G, \cdot_G), (H, \cdot_H)$ be groups.

1. A map $f\colon G \to H$ is called a **group homomorphism** or **homomorphism of groups** if $\forall g, g' \in G\colon f(g \cdot_G g') = f(g) \cdot_H f(g')$.

2. A group homomorphism $f\colon G \to H$ which is also bijective is called a **group isomorphism**. The groups G and H are then said to be **isomorphic**, denoted $G \cong H$.

3. In the case of $H = G$, a group homomorphism $f\colon G \to G$ is called a **group endomorphism**.

4. A group endomorphism $f\colon G \to G$ which is also bijective is called a **group automorphism**.

If the concept of interest is (implicitly) known, it is common to drop the word for it in the description of homomorphisms, i. e. we would refer to a group isomorphism just as an isomorphism if it is obvious that we want to talk about the structure of a group. It is then implicitly understood to preserve this structure.

Remark 2.1.5.

1. It can be proven that for any homomorphism $f\colon G \to H$ between two groups G, H we have: $f(e_G) = e_H$ and $\forall g \in G\colon f\left(g^{-1}\right) = \left(f\left(g\right)\right)^{-1}$, i. e. neutral elements and inverse elements are mapped onto each other, respectively.

2. A well-known result in group theory (and many other algebraic areas of study) is that the composition of two homomorphisms (isomorphisms) $f\colon G \to H, g\colon H \to J$ for groups G, H, J, i. e. $g \circ f\colon G \to J$, is again a homomorphism (isomorphism).

3. Automorphisms of a group G form a group under composition. This is called the **automorphism group** of G, denoted $\mathrm{Aut}(G)$.

Isomorphisms between two groups can be thought of as a re-labelling of the elements of the group such that it is compatible with group multiplication. They are of particular interest since the two groups can be viewed as indistinguishable by methods of group theory, i. e. they share the same group theoretic properties. In particular, they are of the same order. It is often useful to study groups "up to isomorphism" which means that we do not to distinguish between groups which are isomorphic. This is especially useful in the classification of groups.

We want to present some explanatory examples of group homomorphism to give more insight into this notion.

Example 2.1.6.

1. For any group G, the **identity map** $\mathrm{id}\colon G \to G, g \mapsto g$ is a group automorphism of G.

2. For any abelian group G, the map $f := (\cdot)^{-1}\colon G \to G, g \mapsto g^{-1}$ is an involutory group automorphism of G, i. e. $f^2 = \mathrm{id}$, since $\forall g \in G\colon f(f(g)) = f\left(g^{-1}\right) = \left(g^{-1}\right)^{-1} = g$. In the general case, this map will not be a homomorphism since we have $(gh)^{-1} = h^{-1}g^{-1}$ for $g, h \in G$.

3. For any group G and group H, the map $\varphi\colon G \to H, g \mapsto e_H$ is a group homomorphism, the so-called **trivial homomorphism**. It is an isomorphism if both G and H are trivial groups.

4. Let $\mathrm{U}(1) = \{z \in \mathbb{C} \mid |z| = 1\}$ be the group of elements of the complex numbers $\mathbb{C}$ with unit length and let $\mathrm{SO}(2) = \{M \in \mathrm{Mat}(2 \times 2, \mathbb{R}) \mid M^T M = I_2, \det(M) = 1\}$ be orthogonal (2×2)-matrices with determinant 1 where I_2 denotes the identity matrix in 2 dimensions. Then, $\mathrm{U}(1) \cong \mathrm{SO}(2)$ by the isomorphism

$$\mathrm{U}(1) \ni z = \cos(\theta) + i\sin(\theta) \mapsto \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \in \mathrm{SO}(2)$$

for $\theta \in [0, 2\pi)$.

5. The **canonical projection** $\pi\colon \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}, x \mapsto [x]$ is a surjective homomorphism of abelian groups and maps each integer onto its equivalence class modulo $n \in \mathbb{N}$.

6. Let $g \in G$ be a fixed element of a group G. Then, the map $c_g\colon G \to G, h \mapsto ghg^{-1}$ is an automorphism of G, called **conjugation**.

Since the notion of a group is now established, we want to consider a few concept which generate new groups from already known ones.

When considering groups as sets equipped with a binary operation, one quite naturally might ask whether a certain subset of a group is again a group with respect to the multiplication restricted to this subset. This leads to the concept of *subgroups*. It turns out that homomorphisms of groups naturally yield two subgroups, one of the source and one of the target of the homomorphism. These are known as *kernel* and *image* of this homomorphism. The following definition makes all of these notions more precise.

Definition 2.1.7. Let G, H be groups and let $\phi\colon G \to H$ be a group homomorphism.

1. A subset $U \subseteq G$ is called a **subgroup** of G if
   (SG1) U is non-empty, i. e. $U \neq \emptyset$, or, equivalently, $e_G \in U$,
   (SG2) U is closed under multiplication, i. e. $\forall h, k \in U\colon hk \in U$, and
   (SG3) U is closed under inversion, i. e. $\forall u \in U\colon u^{-1} \in U$.

2. A subgroup $U \subseteq G$ is called **normal** if it is invariant under conjugation, i. e. $\forall u \in U, g \in G\colon c_g(u) := gug^{-1} \in U$.

3. The **image** $\operatorname{im}(\phi) \subseteq H$ of ϕ is defined as $\operatorname{im}(\phi) := \{h \in H \mid \exists\, g \in G\colon \phi(g) = h\}$.

4. The **kernel** $\ker(\phi) \subseteq G$ of ϕ is defined as the pre-image of $\{e_H\} \subseteq H$, i. e. $\ker(\phi) := \{g \in G \mid \phi(g) = e_H\}$.

The following lemma provides more usable conditions whether a subset of a group is a subgroup and gives a few facts about the image and the kernel of a group homomorphism.

Lemma 2.1.8. Let G, H be groups and let $\phi\colon G \to H$ be a group homomorphism.

1. A subset $U \subseteq G$ is a subgroup of G if
   a) U is closed under multiplication, i. e. $\forall g, h \in U\colon gh \in U$, and
   b) U equipped with the restriction of the multiplication of G onto U is again a group.

2. The image $\operatorname{im}(\phi) \subseteq H$ of ϕ is a subgroup of H.

3. The kernel $\ker(\phi) \subseteq G$ of ϕ is a normal subgroup of G.

4. ϕ is surjective if $\operatorname{im}(\phi) = H$.

5. ϕ is injective if $\ker(\phi) = \{e_G\}$.

Even though the notion of a subgroup might be rather intuitive, we nevertheless want to give a few examples.

Example 2.1.9.

1. Every group G has trivially two normal subgroups, namely itself, i. e. $G \subseteq G$, and the **trivial** subgroup $\{e_G\} \subseteq G$.

2. Let $H \subseteq G$ be a subgroup of an abelian group G. Then it is also normal since conjugation by any element $g \in G$ acts trivially, i. e. $c_g(x) = gxg^{-1} = gg^{-1}x = e_G x = x$ for any $x \in H$ (and even any $x \in G$).

3. Let $n \in \mathbb{N}$ be a fixed natural number. Then, integers times n, i. e. $n\mathbb{Z} = \{nz \mid z \in \mathbb{Z}\} \subseteq \mathbb{Z}$, form a normal subgroup of Z. It can be seen as the kernel of the canonical projection $\pi\colon \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$.

4. Let G be a group. The so-called **center** $\mathrm{Z}(G)$ of G, defined by $\mathrm{Z}(G) \coloneqq \{z \in G \mid \forall g \in G\colon zg = gz\} \subseteq G$, is a normal subgroup of G.
   It can be shown that G is abelian if and only if $\mathrm{Z}(G) = G$. G is called **centerless** if $\mathrm{Z} = \{e_G\}$.

5. The group of conjugations c_g of a group G is a subgroup of the automorphism group of G and is called the **inner automorphism group**, denoted $\mathrm{Inn}(G)$, i. e. $\mathrm{Inn}(G) = \{c_g\colon G \to G, x \mapsto gxg^{-1} \mid g \in G\} \subseteq \mathrm{Aut}(G)$.

Normal subgroups, especially in the case of a non-abelian group, can be seen as just some sort of special subgroups but they can be used for much more, e. g. for the construction of *quotient groups* using an equivalence relation which preserves some of the original group structure. One of the prime examples of a quotient group is the well-known group $\mathbb{Z}/n\mathbb{Z}$. Many other examples arise in such fashion. Later on, the symmetry group of projective space, i. e. maps which preserve its structure, will be defined as a quotient group in the algebraic way.

Definition 2.1.10. Let G be a group and $N \subseteq G$ a normal subgroup.

1. The **quotient group** G/N of G and N is defined as the left cosets of N in G, i. e. $G/N \coloneqq \{gN \mid g \in G\}$ where $gN \coloneqq \{gn \mid n \in N\}$.
   Its group multiplication is given by $(gN)(hN) \coloneqq (gh)N$ for $g, h \in G$ and its neutral element is the the coset $N \in G/N$.

2. The **canonical projection** $\pi\colon G \to G/N$ of G onto G/N is given by $\pi(g) \coloneqq gN$. It is a surjective group homomorphism with kernel $\ker(\pi) = N \subseteq G$.

Remark 2.1.11.

1. The multiplication of G/N in the above definition is well-defined, i. e. is independent of the choice of the representatives g, h of the cosets gN, hN, respectively, only because $N \subseteq G$ is a normal subgroup. This can be seen by an equivalent definition of normal subgroups whereby the left and right cosets of a subgroup need to coincide, i. e. $gN = Ng \; \forall g \in G$.

2. The multiplication of G/N can be understood as performing the usual multiplication in G and afterwards setting all factors which involve the normal subgroup $N \in G$ to $e_G \in G$. Thus, the phrase "G modulo N" is also often used.

3. If G and $N \subseteq G$ are finite, we find that the order of G/N is given by $|G/N| = \frac{|G|}{|N|}$.

Example 2.1.12.

1. Let G be a group. Then, $G/G \cong \{e\}$ and $G/\{e_G\} \cong G$.

2. Integers modulo $n \in \mathbb{N}$ are a quotient group of $\mathbb{Z}$ by the normal subgroup $n\mathbb{Z} \subseteq \mathbb{Z}$. This is already indicated by the notation $\mathbb{Z}/n\mathbb{Z}$.

3. Let $\mathrm{GL}(n, \mathbb{R})$ be the invertible $(n \times n)$-matrices over the real numbers and let $\mathrm{SL}(n, \mathbb{R}) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid \det(A) = 1\} \subseteq \mathrm{GL}(n, \mathbb{R})$ be the normal subgroup of invertible $(n \times n)$-matrices over the real numbers with determinant 1, then $\mathrm{GL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{R}) \cong \mathbb{R}^\times = \mathbb{R} \setminus \{0\}$.

4. The group of inner automorphisms $\mathrm{Inn}(G)$ of a group G is isomorphic to the quotient group of G by its center $\mathrm{Z}(G)$, i. e. $\mathrm{Inn}(G) \cong G/\mathrm{Z}(G)$.

One of the most interesting cases where quotient groups are used is in the so-called *isomorphism theorems* by Emmy Noether (1882 - 1935), one of which connects the source, the kernel and the image of a group homomorphism which is often called *homomorphism theorem*. Others consider quotients of quotients or quotients by an intersection of two subgroups. We only present the first one since it is useful in the computation of isomorphisms.

Theorem 2.1.13 (Homomorphism Theorem)**.** Let G, H be groups and let $f\colon G \to H$ be a homomorphism.

Then, the image of f is isomorphic to the quotient of G by the kernel of f, i. e.

$$\mathrm{im}(f) \cong G/\ker(f).$$

In particular, if the homomorphism f is surjective, we find that $H \cong G/\ker(f)$.

Remark 2.1.14. With the homomorphism theorem, the kernel of a homomorphism can be understood as a measure of the failure of the homomorphism being an isomorphism onto its image. This might be brought up by intuition but is made precise by the statement above.

Another interesting concept which will be used later on and gives rise to many new examples of groups from existing ones are so-called *semidirect products* of groups. These are formed by two groups but the multiplication is altered from the diagonal version in the direct product of groups to a multiplication where one of the groups "interferes" in the multiplication of the other group. So it makes it a generalization of the direct product. To make this "interference" precise, we give the following definition.

Definition 2.1.15. Let G, H be groups. Let $\rho\colon H \to \mathrm{Aut}(G)$ be a group homomorphism.

The (outer) **semidirect product** $G \rtimes_\rho H$ of G and H is given as a set by the Cartesian product $G \times H$ and its multiplication is given by

$$(g_1, h_1) \cdot_{G \rtimes_\rho H} (g_2, h_2) \coloneqq (g_1 \cdot_G \rho(h_1)(g_2), h_1 \cdot_H h_2)$$

for $g_1, g_2 \in G$ and $h_1, h_2 \in H$.

Example 2.1.16.

1. If the homomorphism ρ in the above definition is taken to be trivial, i. e. $\rho\colon H \to \mathrm{Aut}(G), h \mapsto \mathrm{id}_G$, the resulting semidirect product $G \rtimes_\rho H$ will be the direct product $G \times H$.

2. For $G = \{e\}$ or $H = \{e\}$, the only admissible homomorphism ρ is the trivial one. We find that $G \rtimes_\rho H \cong H$ in the case of $G = \{e\}$ and $G \rtimes_\rho H \cong G$ in the case of $H = \{e\}$. These are called *trivial* semidirect products.

3. The so-called *inner* semidirect product is constructed by two subgroups $N, U \subseteq G$ of a group G where $N \subseteq G$ is taken to be normal. Furthermore, we require G to be the product of both subgroups, i. e. $G = NU$, and $N \cap U = \{e_G\}$. If we then take $\rho\colon U \to \mathrm{Aut}(N), u \mapsto (c_u\colon N \to N, n \mapsto unu^{-1})$ to be the homomorphism in the definition above, we get a semidirect product $N \rtimes_\rho U$ which turns out to be isomorphic to G itself.

4. Let $\mathrm{O}(n)$ be the group of orthogonal real matrices acting on the n-dimensional real vector space $\mathbb{R}^n$ and let $\rho\colon \mathrm{O}(n) \to \mathrm{Aut}(\mathbb{R}^n)$ be the inclusion map by the identification as automorphisms by the natural action of $\mathrm{O}(n)$ on $\mathbb{R}^n$ by matrix-vector multiplication.
   The isometries of Euclidean space $\mathbb{R}^n$ are given by the semidirect product $\mathbb{R}^n \rtimes_\rho \mathrm{O}(n)$ which consists of translations and rotations. The multiplication in $\mathbb{R}^n \rtimes_\rho \mathrm{O}(n)$ is thus given by $(t_1, O_1)(t_2, O_2) \coloneqq (t_1 + O_1 t_2, O_1 O_2)$ for $t_1, t_2 \in \mathbb{R}^n, O_1, O_2 \in \mathrm{O}(n)$.

Often, groups can be hard to handle since they can be quite abstract. In a similar fashion, in linear algebra it is often useful to work with a basis of a vector space which spans the vector space instead of abstract vectors in the vector space. If the vector space is finite-dimensional, it is enough to work with only a finite number of vectors. Almost every aspect of the theory of vector spaces can be condensed to aspects concerning only the basis vectors. For example, the description of a linear map can be done in terms of matrices with respect to this basis which is much more concrete.

A similar construction can be also be done in the case of groups. This is known as the *generation* of a group from a subset of the group and is defined as follows.

Definition 2.1.17. Let G be a group and let $M \subseteq G$ be a subset.

1. The subgroup $\langle M \rangle := \{g_1 g_2 \dots g_n \in G \mid n \in \mathbb{N}_0, g_1, \dots, g_n \in M \cup M^{-1}\} \subseteq G$ is called the **subgroup generated by** M where $M^{-1} = \{m^{-1} \mid m \in M\}$ and $g_1 \dots g_n := e_G$ for $n = 0$.
   M is then called a **generating set** of G. Its elements are called **generators**.
2. G is called **finitely generated** if there exists some subset $M \subseteq G$ with $\langle M \rangle = G$ which is finite, i. e. $|M| < \infty$.
3. G is called **cyclic** if it is generated by one element, i. e. $G = \langle g \rangle = \{g^n \mid n \in \mathbb{Z}\}$.
4. The **order** $o(g)$ of an element $g \in G$ is the smallest $n \in \mathbb{N}$ such that $g^n = e_G$ if it exists, otherwise we define $o(g) = \infty$.

Remark 2.1.18.

1. Notice that a generating set M of a group G is not unique since we could for example also use M^{-1} because $(M^{-1})^{-1} = M$.
2. It is obvious that any cyclic group is necessarily abelian since we have $g^m g^k = g^{m+k} = g^{k+m} = g^k g^m$.
3. It can be shown that in a finite cyclic group G any element g with order $o(g) = |G|$ is a generator of G.

Example 2.1.19.

1. Since every group is a subgroup of itself, it is also generated by itself.
2. The integers $\mathbb{Z}$ are a cyclic group with $\mathbb{Z} = \langle 1 \rangle$ since:
   $\mathbb{N} \ni n = \underbrace{1 + \dots + 1}_{n\times}$ and $-n = \underbrace{(-1) + \dots + (-1)}_{n\times}$.
3. For any $n \in \mathbb{N}$, the group $\mathbb{Z}/n\mathbb{Z}$ is cyclic with generator for example the equivalence class of 1, i. e. $\langle [1] \rangle = \mathbb{Z}/n\mathbb{Z}$.
   It can be shown that an infinite cyclic group is isomorphic to $\mathbb{Z}$ and a finite cyclic group is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ for some $n \in \mathbb{N}$. Hence, these two are the prime examples of cyclic groups.
4. The symmetry group of a regular n-gon, the so-called *dihedral group* D_n, is finitely generated by two elements, a reflection s with $o(s) = 2$ and a rotation r with $o(r) = n$ and $srs = r^{-1}$.
   This can be written in terms of a so-called **presentation** with generators and relations on these generators, i. e. $D_n = \langle r, s \mid o(r) = n, o(s) = 2, srs = r^{-1} \rangle$.

As we have seen in an example above, groups can also act on sets such as the group $\mathrm{GL}(n, \mathbb{R})$ acts on the vector space $\mathbb{R}^n$ by matrix-vector multiplication. This action also respects the structure of the group which means that it is irrelevant if one group element acts on a vector and afterwards another group element acts on the previous result or we first use the group multiplication to multiply both group elements and then act on the vector. As a generalization for arbitrary groups acting on a set we define a *group action* as follows.

Definition 2.1.20. Let G be a group and X be a set.

1. A (left) **group action** $\triangleright\colon G \times X \to X, (g, x) \mapsto g \triangleright x$ is a map which satisfies the following conditions:
   a) The natural element $e_G \in G$ acts trivially, i. e. $\forall x \in X\colon e_G \triangleright x = x$.
   b) It respects group multiplication, i. e. $\forall g, h \in G, x \in X\colon g \triangleright (h \triangleright x) = (gh) \triangleright x$.

   X is then called a **G-set**.
2. An action $\triangleright\colon G \times X \to X$ is called **transitive** if $X \neq \emptyset$ and $\forall\,(x, y) \in X \times X \;\exists\, g \in G$ such that $g \triangleright x = y$.
3. An element $x \in X$ is called **fixed point** of the action or **G-invariant** if $\forall g \in G\colon g \triangleright x = x$. The set of all G-invariants is denoted X^G.
4. The **orbit** $G \triangleright x$ of an element $x \in X$ is the set $G \triangleright x \coloneqq \{g \triangleright x \mid g \in G\} \subseteq X$.
5. The **stabilizer** G_x of an element $x \in X$ is the subgroup $G_x \coloneqq \{g \in G \mid g \triangleright x = x\} \subseteq G$.

Remark 2.1.21.

1. There is also a notion of a **right group action** of a group G on a set X which is defined as a map $\triangleleft\colon X \times G \to X, (x, g) \mapsto x \triangleleft g$ satisfying $x \triangleleft e_G = x$ and $\forall g, h \in G\colon (x \triangleleft g) \triangleleft h = x \triangleleft (gh)$.
   A left group action $\triangleright\colon G \times X \to X$ can be changed into a right group action $\triangleleft\colon X \times G \to X$ (or vice-versa) by applying an inverse to the group elements, i. e. for $x \in X, g \in G$ we define $x \triangleleft g \coloneqq g^{-1} \triangleright x$. We will refer to left group actions as group actions.
2. If we fix an element $g \in G$ of a group G, we get from a group action $\triangleright\colon G \times X \to X, x \mapsto g \triangleright x$ a map $\triangleright_g\colon X \to X$ which is bijective since its inverse is given by $\triangleright_{g^{-1}}\colon X \to X$. Therefore, the map $g \mapsto \triangleright_g$ is a group homomorphism from G to $\mathrm{Sym}(X)$, i. e. the group of all bijections of X to X which can be understood as the permutation group on X.
   Conversely, we may also define a group action by a group homomorphism from G to $\mathrm{Sym}(X)$.

We have already motivated that matrix-vector multiplication is a known example. We want to give a few more common examples which arise in various contexts.

Example 2.1.22.

1. Let X be any set. Then, there is a group action for arbitrary groups G on X, the **trivial action**, which is defined by $g \triangleright x = x$ for all $g \in G$ and $x \in X$.

2. For any group G, we can define an action of G on G by left multiplication, i. e. $g \triangleright h := gh$ for $g, h \in G$.
   This property is assured by the associativity of the multiplication and the properties of the neutral element. This action is also transitive since inverse elements exist for every element of G, i. e. for $(g, h) \in G \times G$ we have $(hg^{-1}) \triangleright g = hg^{-1}g = h$.

3. In a similar fashion, for any group G, we can define a right action by right multiplication, i. e. $g \triangleleft h := gh$ for $g, h \in G$ where it is understood that h is acting on g.

4. For any group G, conjugation defines a group action of G on itself, i. e. for $g, h \in G$ we have $g \triangleright h = c_g(h) = ghg^{-1}$.

5. Let $X = \mathbb{R}^n, n \in \mathbb{N}$ and $G = \mathrm{GL}(n, \mathbb{R})$, then $\triangleright \colon \mathrm{GL}(n, \mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n, (M, x) \mapsto Mx$, i. e. matrix-vector multiplication, is a group action of $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$.

This shows that a group action can take many different forms. We will use this concept later to describe the action of a projective transformation on a quadric.

2.2 Fields

After studying the theory of groups we are now fully equipped to study sets with not only one but two binary operations which are compatible. This yields a lot more structure and gives rise to the notions of *rings* and *fields*. These are vital for the definition of modules over some ring or vector spaces over some field which arise in almost all areas of physics and mathematics.
One of the prime examples of a ring is the ring of polynomials over the real numbers, a field, which can be used to construct the field of complex numbers, a so-called *field extensions* of the real numbers. This concept will be made more precise which leads to *Galois theory* where transformations like the complex conjugation play an important role. We also want to specialize to the case of finite fields which will be the starting point of the construction of our physical theory.

2.2.1 From Rings to Fields

At first, we introduce the concept of a ring which can be thought of as a generalization of the integers $\mathbb{Z}$.

Definition 2.2.1.

1. A triple $(R, +, \cdot)$ with a set R and two binary operations $+, \cdot \colon R \times R \to R$ is called a **(unital) ring** if the following axioms are satisfied:

   (R1) $(R, +)$ forms an abelian group. Its neutral element is called **zero element**, denoted $0 \in R$. The operation $+$ is called **addition**. The additive inverse of an element $r \in R$ is denoted $-r \in R$.

   (R2) $(R, \cdot)$ is a monoid, i. e. $\cdot \colon R \times R \to R$ is associative and there exists some neutral element $1 \in R$ for the binary operation $\cdot$. The operation $\cdot$ is called **multiplication**.

   (R3) Multiplication is distributive with respect to addition, i. e. the two laws of (left and right) distributivity are fulfilled:

   $$\forall a, b, c \in R \colon a \cdot (b + c) = (a \cdot b) + (a \cdot c) \text{ and } (a + b) \cdot c = (a \cdot c) + (b \cdot c).$$

2. A ring $(R, +, \cdot)$ is called **commutative** if its multiplication is commutative, i. e. $\forall a, b \in R \colon a \cdot b = b \cdot a$.

3. An element $r \in R$ is called **unit** if there exists an multiplicative inverse, denoted r^{-1}, i. e. $\exists\, r^{-1} \in R \colon r \cdot r^{-1} = r^{-1} \cdot r = 1$. The **group of units** is denoted $(R^\times, \cdot)$ with neutral element $1 \in R$.

Remark 2.2.2.

1. There is also a definition of a ring without the condition of needing to have a neutral element for multiplication, This structure is more general but much less intuitive to use. Thus, we will only use the former definition with neutral element.

2. It is common to drop the symbol of multiplication with the caveat that multiplication is 'stronger' than addition, meaning that it is understood to first multiply and then add any expressions. The laws of distributivity then call for the compatibility of these two operations.

3. In ring theory, several notations of the addition and multiplication of the integers $\mathbb{Z}$ are adopted. For example, for a natural number $n \in \mathbb{N}$ and an element $r \in R$ of a ring R we define $n \cdot r := \underbrace{r + \cdots + r}_{n\times}$ and $r^n := \underbrace{r \cdots r}_{n\times}$.

4. It can be shown that in any ring R we have $0 \cdot r = r \cdot 0 = 0$ for every $r \in R$.

As the definition shows, rings have a lot more structure than groups since there isn't merely one operation but two which have to be compatible. There are several examples of how to construct new rings form already known ones.

Example 2.2.3.

1. The set $\{0\}$ becomes a commutative ring, the so-called **trivial ring**, when equipped with trivial addition and multiplication. Here, we have that $0 = 1$.

2. The integers $\mathbb{Z}$ form a commutative ring with usual addition and multiplication where $0, 1 \in \mathbb{Z}$ are the neutral elements, respectively. The group of units of $\mathbb{Z}$ is given by $\mathbb{Z}^\times = \{-1, 1\}$.

3. Real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$ and rational numbers $\mathbb{Q}$ form commutative rings with their usual addition and multiplication, respectively. Their group of units is given by all non-zero elements.

4. Given a commutative ring R, there is the ring of polynomials in one variable $R[X]$ with coefficients in R which is also commutative. The neutral element of multiplication is given by the constant polynomial $1 \cdot X^0 \in R[X]$. Its units are the constant polynomials $R^\times X^0 \subseteq R[X]$.

5. The polynomial ring $R[X_1, \dots, X_n]$ in multiple variables with coefficients in the commutative ring R is iteratively defined as the polynomial ring of the commutative ring $R[X_1, \dots, X_{n-1}]$, i. e. $R[X_1, \dots, X_n] := R[X_1, \dots, X_{n-1}][X_n]$.

6. Let $n \in \mathbb{N}$. Integers modulo n, $\mathbb{Z}/n\mathbb{Z}$, form a commutative ring with addition and multiplication performed modulo n. The neutral element of multiplication is the equivalence class of $1 \in \mathbb{Z}$ and the group of units is given by $(\mathbb{Z}/n\mathbb{Z})^\times = \{[m] \mid m \in \mathbb{Z}, \gcd(m, n) = 1\} \subseteq \mathbb{Z}/n\mathbb{Z}$. If $n \in \mathbb{N}$ is prime, it can be proven that $(\mathbb{Z}/n\mathbb{Z})^\times = \mathbb{Z}/n\mathbb{Z} \setminus \{[0]\}$.

As in the case of groups, it is not only useful to study the objects of ring theory but also maps between the objects which preserve their structure. In the case of rings, these are called *ring homomorphisms* and are defined as follows.

Definition 2.2.4. Let $(R, +_R, \cdot_R), (S, +_S, \cdot_S)$ be rings.

1. A map $\varphi\colon R \to S$ is called a **(unital) ring homomorphism** if it is a group homomorphism of the groups $(R, +_R)$ and $(S, +_S)$ and preserves multiplication, i. e. satisfies the following conditions:
   a) $\forall r, r' \in R\colon \varphi(r \cdot_R r') = \varphi(r) \cdot_S \varphi(r')$.
   b) $\varphi(1_R) = 1_S$.

2. A ring homomorphism $\varphi\colon R \to S$ is called a **ring isomorphism** if it is bijective.

3. A ring homomorphism $\varphi\colon R \to R$ from R to itself is called a **ring endomorphism**. If it is also bijective, it is called a **ring automorphism**.

4. The **kernel** $\ker(\varphi)$ of a ring homomorphism $\varphi\colon R \to S$ is defined as $\ker(\varphi) := \{r \in R \mid \varphi(r) = 0_S\} \subseteq R$.

5. The **image** $\operatorname{im}(\varphi)$ of a ring homomorphism $\varphi\colon R \to S$ is defined as $\operatorname{im}(\varphi) := \{s \in S \mid \exists\, r \in R\colon \varphi(r) = s\} \subseteq S$.

Remark 2.2.5.

1. Since there exists also the concept of non-unital rings, there is also a notion of ring homomorphism which are not unital, i. e. do not map the neutral elements of multiplication onto each other. Since we will be concerned with fields which will turn out to be special cases of unital rings, we do not want to discuss this further.

2. It can be proven that a ring homomorphism $\varphi\colon R \to S$ is injective if $\ker(\varphi) = \{0\} \subseteq R$, and it is surjective if $\mathrm{im}(\varphi) = S$.

Example 2.2.6.

1. For every ring R, the identity map $\mathrm{id}\colon R \to R, r \mapsto r$ is a ring automorphism.

2. For every ring R, we can construct a canonical ring homomorphism $\varphi\colon \mathbb{Z} \to R$ by

$$\forall n \in \mathbb{N}_0\colon \varphi(n) = n \cdot 1_R = \underbrace{1_R + \cdots + 1_R}_{n\times} \text{ and } \varphi(-n) = -\varphi(n)$$

   where we define $\varphi(0) \coloneqq 0_R$.
   It can be shown that the kernel of φ is given by $m\mathbb{Z}$ for some $m \in \mathbb{N}_0$. m is then called the **characteristic** of R, usually denoted $\mathrm{char}(R)$.

3. Let $R = \mathbb{Z}/p\mathbb{Z}$ for a prime $p \in \mathbb{N}$ in the above example. Then, $\ker(\varphi) = p\mathbb{Z}$ since $p \cdot [1] = [p] = [0]$ and $n \cdot [1] = [n] \neq [0]$ for any $n \in \mathbb{N}$ with $p > n$. Thus, its characteristic is p.

4. Let $R[X]$ be the polynomial ring over some commutative ring R and let $a \in R$ be a fixed element. The **evaluation map** $\mathrm{ev}_a\colon R[X] \to R, f \mapsto \mathrm{ev}_a(f) \coloneqq f(a)$ is a ring homomorphism for every fixed $a \in R$.

5. Let R be a commutative ring with prime characteristic p. It can be shown that the **Frobenius homomorphism** $\phi_p\colon R \to R, r \mapsto r^p$ is a ring endomorphism.
   If $R = \mathbb{Z}/p\mathbb{Z}$ in the example above, then the Frobenius homomorphism $\phi_p\colon \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}, x \mapsto x^p$ is the identity on $\mathbb{Z}/p\mathbb{Z}$ by Fermat's little theorem.

Ring homomorphism and, in particular, the Frobenius homomorphism will become important later in the Galois theory of finite fields which can be thought of as extensions of $\mathbb{Z}/p\mathbb{Z}$ for some prime $p \in \mathbb{N}$ in some sense.

Just as in the case of groups, it will prove to be useful to study subsets of some ring R which form a subgroup of the group $(R, +)$ and are also compatible with multiplication. We could on the one hand ask for this subset to also form a ring where multiplication of two elements of the subset is again in the subset and 1_R is in the subset as well which leads to the notion of *subrings* which we do not want to explore further.
We could on the other hand require that this subset 'absorbs' multiplication of any element of the ring, that is multiplication of an element of this subset with any element of the ring be in the subset again. Since multiplication is, in general, non-commutative,

multiplication form the right and from the left have to be distinguished with care. These subsets are called *(left/right) ideals* and will allow us to construct quotients of rings by ideals as we did with groups and normal subgroups. This notion simplifies in the case of commutative rings.

Definition 2.2.7. Let $(R, +, \cdot)$ be a ring.

1. A subset $S \subseteq R$ is called a (unital) **subring** if it forms a (unital) ring with respect to the restriction of the operations of R.
2. A subset $I \subseteq R$ is called a **left ideal** if
   a) $(I, +)$ is a subgroup of $(R, +)$, and
   b) I absorbs multiplication of R from the left, i. e. $\forall i \in I, r \in R\colon r \cdot i \in I$.
3. A subset $I \subseteq R$ is called a **right ideal** if
   a) $(I, +)$ is a subgroup of $(R, +)$, and
   b) I absorbs multiplication of R from the right, i. e. $\forall i \in I, r \in R\colon i \cdot r \in I$.
4. A subset $I \subseteq R$ is called a (two-sided) **ideal** if it is both a left and a right ideal.
5. For $x \in R$ we define the **principal left/right ideals** generated by x by Rx and xR, respectively, i. e. generated by one element $x \in R$. The (two-sided) **principal ideal** generated by x, RxR, is denoted (x).

Remark 2.2.8.

1. If the requirement for a subring having to contain the multiplicative neutral element is dropped, an ideal also forms a subring.
2. In the case of commutative rings, the definition of right, left and two-sided ideals coincide.
3. Ideals can be thought of as generalisations of the subset of even integers $2\mathbb{Z}$ in the ring $\mathbb{Z}$ where we find that multiplication with any element of $\mathbb{Z}$ yields again an even integer and the sum of two even integers is again even.
4. Note that if an ideal $I \subseteq R$ contains the neutral element of multiplication of the ring R, $1 \in I$, we find that $r \cdot 1 = r \in I$ for all $r \in R$. This means that $R \subseteq I \subseteq R$, so $R = I$.

Example 2.2.9.

1. For any ring R, the set $\{0_R\} \subseteq R$ forms an ideal, the **zero ideal**.
2. For any ring R, the ring itself forms an ideal which is generated by 1_R, i. e. $(1_R) = R$. (Left, right, two-sided) ideals which are strictly smaller than (1_R) are called **proper ideals**.

3. It can be shown that the sum of two left (right) ideals is again a left (right) ideal.

4. Let $n \in \mathbb{N}$. The multiples of n, $n\mathbb{Z} \subseteq \mathbb{Z}$, form an ideal since this factor n is preserved by addition of two multiples of n and by multiplication with any other integer.

As indicated above, with ideals we can construct *quotient rings* whose construction can be compared to the construction of quotient groups by normal subgroups or the construction of integers modulo n. In the latter, we set multiples of n, which are therefore contained in $n\mathbb{Z}$, to zero and do calculations based on this procedure. The construction of quotient rings can be seen in a similar manner where we set element of the ideal to zero in any calculations. To make this more precise, we come to the following definition.

Definition 2.2.10. Let R be a ring and $I \subseteq R$ a two-sided ideal.

1. On R, we define an equivalence relation $\sim$ by $a \sim b \iff a - b \in I$ for $a, b \in R$. The equivalence class of $r \in R$ is denoted $[r]$ or $r + I$.

2. The **quotient ring** R/I of R modulo I is defined as the set of equivalence classes of the equivalence relation $\sim$ on R, i. e. $R/I = \{r + I \mid r \in R\}$.
   Its addition is given by $[a] + [b] = (a + I) + (b + I) = (a + b) + I = [a + b]$ with zero element $[0] = 0 + I = I$. Multiplication is given by $[a] \cdot [b] = (a + I) \cdot (b + I) = (a \cdot b) + I = [a \cdot b]$ with neutral element $[1] = 1 + I$.

3. The surjective ring homomorphism $\pi \colon R \to R/I, r \mapsto r + I$ is called the **canonical homomorphism** of R/I.

Remark 2.2.11. Multiplication and addition in the definition of a quotient ring are well-defined since $I \subseteq R$ is ought to be a two-sided ideal and, thus, absorbs multiplication from the left and from the right.

Example 2.2.12.

1. For any ring R, for $I = \{0_R\} \subseteq R$ we find that R/I is naturally isomorphic to R. For $I = R$, it can be shown that $R/R \cong \{0\}$, the trivial ring. This can be thought of as setting every element of R to zero.

2. For any $n \in \mathbb{N}$, integers modulo n, $\mathbb{Z}/n\mathbb{Z}$, are a quotient ring of $\mathbb{Z}$ by the ideal $n\mathbb{Z}$.

3. Let $R = \mathbb{R}[X]$ be the polynomials in one variable with real coefficients and let $I = (X^2 + 1)$ be the ideal generated by the polynomial $p = X^2 + 1 \in \mathbb{R}[X]$. The quotient ring $\mathbb{R}[X]/(X^2 + 1)$ is isomorphic to the complex numbers $\mathbb{C}$ by the map that evaluates a polynomial $f \in \mathbb{R}[X]$ at $i \in \mathbb{C}$ since the minimal polynomial with real coefficients of the imaginary unit $i \in \mathbb{C}$ is given by $X^2 + 1$. This example is an example of what we will call a *field extension* which will be made more precise later.

As in the case of groups, there are several isomorphism theorems which use quotient rings, named after Emmy Noether. We want to mention one of them which is known as the homomorphism theorem and is in complete analogy with the one for groups. This can be used to find isomorphism between (quotient) rings by using a ring homomorphism.

Theorem 2.2.13 (Homomorphism Theorem)**.** Let R, S be rings and $\varphi\colon R \to S$ be a ring homomorphism.

Then, $\ker(\varphi) \subseteq R$ is an ideal and

$$\mathrm{im}(\varphi) \cong R/\ker(\varphi).$$

In particular, if φ is surjective, i. e. $\mathrm{im}(\varphi) = S$, we find that $S \cong R/\ker(\varphi)$.

As we have seen in example 2.2.3, the well-known rational, real and complex numbers form commutative rings with the special feature that every non-zero element is a unit and $1 \neq 0$. These are special cases of a wider notion, called *fields*. These are commonly used in various definitions in mathematics and, especially, physics. They also build the foundation of the definition of vector spaces over some field and are defined as follows.

Definition 2.2.14.

1. A commutative ring $(\mathbb{F}, +, \cdot)$ is called a **field** if $1 \neq 0$ and its group of units is given by $\mathbb{F}^\times = \mathbb{F} \setminus \{0\}$, i. e. every non-zero element is invertible.
   If the set $\mathbb{F}$ is finite, i. e. $|\mathbb{F}| < \infty$, it is called **finite field**.

2. The **characteristic** of a field $\mathbb{F}$ is the characteristic of it as a ring.

3. A subring $k \subseteq \mathbb{F}$ is called a **subfield** if it forms a field with respect to the restriction of addition and multiplication of $\mathbb{F}$ onto this subset.

4. A field $\mathbb{F}$ is called **prime field** if it has no proper, i. e. strictly smaller, subfields.

Remark 2.2.15.

1. If the condition of the multiplication of a field to be commutative is dropped and the requirement of having an inverse is adopted to the existence of left- and right-inverses, we get the definition of a **skew-field**. *Wedderburn's little theorem* states that every finite skew-field is actually a field. Since we are primarily interested in a finite approach, we will see that a finite skew-field suffices which is then actually just a finite field. Thus, we won't study the concept of skew-fields further.

2. It can be shown that the characteristic of a field is either 0 or a prime number $p \in \mathbb{N}$. This characteristic carries over to its subfields, meaning that it can only contain subfields of said characteristic.

3. Prime fields can be thought of as the 'smallest' fields of their characteristic. Therefore, every other field of this characteristic contains a prime field as a subfield or is isomorphic to it.

4. The ideals of a field regarded as a ring are rather trivial since they can only be the trivial ideal or the whole field.

Example 2.2.16.

1. The trivial ring $\{0\}$ does not form a field since it does not meet the requirement that $1 \neq 0$.

2. Rational numbers $\mathbb{Q}$, real numbers $\mathbb{R}$ and complex numbers $\mathbb{C}$ each form a field with the inclusions as (proper) subfields $\mathbb{Q} \subsetneq \mathbb{R} \subsetneq \mathbb{C}$. They all have characteristic 0. It can be shown that $\mathbb{Q}$ is also a prime field whereas the other two are, obviously, not.

3. Let $p \in \mathbb{N}$ be prime. $\mathbb{Z}/p\mathbb{Z}$, that is integers modulo p, form a field with its usual addition and multiplication modulo p. They form a prime field of characteristic p, usually denoted $\mathbb{F}_p$.
   It can be proven that $\mathbb{Z}/n\mathbb{Z}$ for $n \in \mathbb{N}$ forms a field if and only if n is prime. The proof uses the fact that prime numbers have no non-trivial divisors since their prime factorization is given, tautological, by $n = n$.

As per usual, we are not only interested in the objects of this new concept but also in maps which preserve the structure of the objects which are herein called *field homomorphism*.

Definition 2.2.17. Let $\mathbb{F}, \mathbb{K}$ be fields.

1. A (unital) ring homomorphism $f: \mathbb{F} \to \mathbb{K}$ between the two field $\mathbb{F}, \mathbb{K}$ is called a **field homomorphism**. If it is also bijective, it is called a **field isomorphism**.

2. For $\mathbb{F} = \mathbb{K}$, a field homomorphism is called **field endomorphism** and a field isomorphism is called **field automorphism**.

3. The **kernel** $\ker(f) \subset \mathbb{F}$ of a field homomorphism $f: \mathbb{F} \to \mathbb{K}$ is the kernel of f as a ring homomorphism.

4. The **image** $\operatorname{im}(f) \subset \mathbb{K}$ of a field homomorphism $f: \mathbb{F} \to \mathbb{K}$ is the image of f as a ring homomorphism.

Remark 2.2.18.

1. It can be shown that every field homomorphism is injective and, thus, its kernel is trivial.
   This comes from the fact that the kernel of a ring homomorphism is an ideal and the only ideals of a field are the trivial ideal and the field itself. Since $1 \neq 0$ in a field, the kernel does not contain the neutral element of multiplication and, thus, cannot be the whole field.

2. The image $\operatorname{im}(f) \subseteq \mathbb{K}$ of a field homomorphism $f: \mathbb{F} \to \mathbb{K}$ is a subfield of $\mathbb{K}$.

Example 2.2.19.

1. The identity map $\mathrm{id}\colon \mathbb{F} \to \mathbb{F}, x \mapsto x$ for any field $\mathbb{F}$ is a field automorphism.

2. The inclusion map $\mathbb{Q} \hookrightarrow \mathbb{R}, q \mapsto q$ is a field homomorphism. Its image is $\mathbb{Q} \subsetneq \mathbb{R}$ as a subfield. This can be done with every subfield of a field.

3. **Complex conjugation** $\bar{\cdot}\colon \mathbb{C} \to \mathbb{C}, z = x+iy \mapsto \overline{z} = x-iy$ is a field automorphism of the complex numbers $\mathbb{C}$ which leaves the subfield $\mathbb{R} \subsetneq \mathbb{C}$ invariant, i. e. $\overline{x} = x, \forall x \in \mathbb{R} \subseteq \mathbb{C}$.

We will see that the concept of field homomorphisms is closely related to the concept of field extensions by means of Galois theory.

2.2.2 Field Extensions and Galois Theory

We have seen that fields, in particular prime fields, can be contained in other fields. In the last example of 2.2.19 we came across such a constellation, that is the field of real numbers is contained in the field of complex numbers as a subfield and the complex numbers can be seen as a two-dimensional vector space of the real numbers. As already mentioned, this is an example of a *field extension*.

But there is more to the last example. It shows that complex conjugation as a field automorphism of the complex numbers respects the structure of this field extension, meaning that it leaves the subfield from which the field extension is performed invariant. This is starting point of a much richer theory, called *Galois theory*, named after Évariste Galois (1811 - 1832), which is concerned with field extensions and automorphism groups of these extensions, known as *Galois groups*.

At first, we want to provide a definition of field extensions.

Definition 2.2.20.

1. A **field extension** $\mathbb{L}/\mathbb{K}$ is a pair of fields $\mathbb{L}, \mathbb{K}$ such that $\mathbb{K} \subseteq \mathbb{L}$ is a subfield. $\mathbb{L}$ is then called **extension field** of $\mathbb{K}$ and $\mathbb{K}$ is called **base field**.

2. If $\mathbb{K} \subseteq \mathbb{F}$ and $\mathbb{F} \subseteq \mathbb{L}$ are field extensions, then $\mathbb{F}$ is called **intermediate field** of the field extension $\mathbb{L}/\mathbb{K}$.

3. The dimension $\dim_{\mathbb{K}}(\mathbb{L})$ of an extension field $\mathbb{L}$ of $\mathbb{K}$ as a vector space over $\mathbb{K}$ is called **degree** of the field extension, denoted $[\mathbb{L} : \mathbb{K}] = \dim_{\mathbb{K}}(\mathbb{L})$.

4. A field extension $\mathbb{L}/\mathbb{K}$ is called **finite** if its degree is finite, i. e. $[\mathbb{L} : \mathbb{K}] < \infty$. Otherwise it is called **infinite**. If the degree $[\mathbb{L} : \mathbb{K}] = 2$, the field extension is called **quadratic**.

One of the most important examples of a field extension is a field extension by adjoining a subset of an extension field to the base field as is the case for the complex numbers as an extension of the real numbers and adjoining the set $\{i\}$. This procedure can be generalized as follows.

Definition 2.2.21. Let $\mathbb{L}/\mathbb{K}$ be a field extension and $S \subseteq \mathbb{L}$ be a subset.

1. The **intermediate field generated by** S is the smallest subfield of $E \subseteq \mathbb{L}$ which contains S, denoted $E = \mathbb{K}(S)$. E is then said to be built from $\mathbb{K}$ by **adjunction** of S.

2. If $\mathbb{L} = \mathbb{K}(a)$ for some $a \in \mathbb{L}$, then $\mathbb{L}/\mathbb{K}$ is called **simple** and $a \in \mathbb{L}$ is called **primitive element** of the field extension.

Remark 2.2.22.

1. The notation $\mathbb{L}/\mathbb{K}$ is not to be confused with a quotient of some sort. Both concepts can be brought together as shown below but are not a priori connected.

2. If a simple field extension $\mathbb{K}(a)/\mathbb{K}$ is of finite degree $[\mathbb{K}(a) : \mathbb{K}] = n$, every element $k \in \mathbb{K}(a)$ can be written as $k = k_0 + k_1 a + \cdots + k_{n-1} a^{n-1}$. Therefore, $\mathbb{K}(a)$ can be seen as an n-dimensional vector space over $\mathbb{K}$ with basis $\{1, a, \ldots, a^{n-1}\}$.

3. Adjoining elements to a base field is especially interesting when considering them as roots of some irreducible polynomial with coefficients in the base field. In the case of complex numbers, this is achieved by the polynomial $X^2 + 1 \in \mathbb{R}[X]$.
   It can be shown that for any irreducible polynomial $p \in \mathbb{K}[X]$, i. e. it has no proper and non-trivial sub-factors, for any field $\mathbb{K}$ the quotient ring $\mathbb{L} := \mathbb{K}[X]/(p)$ is a field and the field extension $\mathbb{L}/\mathbb{K}$ is simple. Its primitive element is given by the equivalence class $[X] \in \mathbb{K}[X]/(p)$.

4. The above construction can be done similarly for any subset of the polynomial ring of some field. This concept then leads to a so-called **splitting field** where any polynomial of this subset factors into linear factors and the extension field is generated by the roots of those polynomials.

Example 2.2.23.

1. The field extension $\mathbb{R}/\mathbb{Q}$ is infinite since $\mathbb{Q}$ is countably infinite and $\mathbb{R}$ is uncountable. If it were finite, $\mathbb{R}$ would be an n-dimensional vector space over $\mathbb{Q}$ and, thus, also countably infinite.

2. The field extension $\mathbb{C}/\mathbb{R}$ is simple and quadratic. A primitive element of this extension is $i \in \mathbb{C}$.

3. The quotient $(\mathbb{Z}/5\mathbb{Z})[X]/(X^2 - [2])$ is a simple and quadratic extension of the field $\mathbb{Z}/5\mathbb{Z}$ since $[2] \in \mathbb{Z}/5\mathbb{Z}$ is a non-square. The elements of the extension field are given by $a + b\alpha$ for $a, b \in \mathbb{Z}/5\mathbb{Z}$ and with $\alpha^2 = [2]$. This is in complete analogy to the case of $\mathbb{C}/\mathbb{R}$.

One of the main goals of the study of field extensions of some base field $\mathbb{K}$ is their classification up to isomorphism. This means that we are interested in isomorphisms of field extensions which should be field isomorphisms of the extension fields which respects the nature of the extensions, i. e. respects the base field which should be the same for both extensions. These homomorphisms are known as $\mathbb{K}$-*homomorphisms*. A special case are the automorphisms of a field extension which form the so-called *Galois group*. One of the main achievements of Galois theory is a correspondence between subgroups of the Galois group of a so-called Galois extension, which is a special type of field extension, and intermediate fields of this extension.

Definition 2.2.24. Let $\mathbb{L}/\mathbb{K}$ and $\mathbb{F}/\mathbb{K}$ be field extensions.

1. A $\mathbb{K}$**-homomorphism** is a field homomorphism $\phi\colon \mathbb{L} \to \mathbb{F}$ such that $\phi(k) = k$ for all $k \in \mathbb{K}$. The definition is analogous for $\mathbb{K}$**-isomorphism**, $\mathbb{K}$**-endomorphism** and $\mathbb{K}$**-automorphism**.

2. The **Galois group** $\mathrm{Gal}(\mathbb{L}/\mathbb{K})$ of the field extension $\mathbb{L}/\mathbb{K}$ is the group of $\mathbb{K}$-automorphisms of $\mathbb{L}$, i. e. $\mathrm{Gal}(\mathbb{L}/\mathbb{K}) = \{\phi \in \mathrm{Aut}(\mathbb{L}) \mid \forall k \in \mathbb{K}\colon \phi(k) = k\}$.

3. The field extension $\mathbb{L}/\mathbb{K}$ is called **Galois extension** if $\mathbb{K}$ is the fixed field of $\mathrm{Gal}(\mathbb{L}/\mathbb{K})$, i. e. $\mathbb{K} = \{l \in \mathbb{L} \mid \forall \phi \in \mathrm{Gal}(\mathbb{L}/\mathbb{K})\colon \phi(l) = l\}$. It is called **cyclic** if its Galois group is cyclic, i. e. generated by one element.

Remark 2.2.25. As extension fields also form vector spaces over their respective base fields, $\mathbb{K}$-homomorphisms can also be thought of as field homomorphisms which are also $\mathbb{K}$-linear.

A special case of Galois extensions are finite Galois extensions. Here, the following result can be proven.

Proposition 2.2.26. Let $\mathbb{L}/\mathbb{K}$ be a Galois extension of finite degree $[\mathbb{L} : \mathbb{K}] < \infty$.
Then, $|\mathrm{Gal}(\mathbb{L}/\mathbb{K})| = [\mathbb{L} : \mathbb{K}]$.

Example 2.2.27.

1. The Galois group of the trivial extensions $\mathbb{K}/\mathbb{K}$ is the trivial group, i. e. $\mathrm{Gal}(\mathbb{K}/\mathbb{K}) = \{\mathrm{id}_{\mathbb{K}}\}$. Its degree $[\mathbb{K} : \mathbb{K}] = 1 = |\{\mathrm{id}_{\mathbb{K}}\}|$.

2. It can be shown that, if $\mathbb{L}/\mathbb{K}$ is a field extension over the prime field $\mathbb{K}$ of $\mathbb{L}$, the Galois group $\mathrm{Gal}(\mathbb{L}/\mathbb{K}) = \mathrm{Aut}(\mathbb{L})$ is just the group of automorphisms of $\mathbb{L}$.

3. The Galois group of $\mathbb{C}/\mathbb{R}$ is given by $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{\mathrm{id}_{\mathbb{C}}, \bar{\cdot}\}$ where $\bar{\cdot}\colon \mathbb{C} \to \mathbb{C}$ is complex conjugation. The extension $\mathbb{C}/\mathbb{R}$ is cyclic and quadratic and, therefore, its Galois group has two elements by above proposition.

4. If the extension field $\mathbb{L}$ of $\mathbb{K}$ is constructed via an irreducible polynomial with coefficients in the base field or is a splitting field, then the Galois group of this extension acts as a permutation group on the roots of the respective minimal polynomials.
This can be seen in the Galois group of $\mathbb{C}/\mathbb{R}$. Here, $\mathbb{C} \cong \mathbb{R}[X]/(X^2 + 1)$ with the roots $i, -i \in \mathbb{C}$ of $X^2 + 1$. The identity on $\mathbb{C}$ acts as the trivial permutation of i

and $-i$. Complex conjugation interchanges both roots since $\bar{i} = -i$ and vice-versa since $\bar{\bar{\cdot}} = \mathrm{id}_{\mathbb{C}}$. Hence, the Galois group acts as an $\mathbb{R}$-linear permutation on the roots of $X^2 + 1$.

These examples show that the computation of the Galois Group of a field extension can be quite manageable under suitable assumptions.

2.2.3 Finite Fields

In our theory, we later want to use finite fields which are fields with only a finite number of elements. We have already seen a few examples of them and we will see that the structure of finite fields and their classification up to isomorphism is remarkably simple.

We start with some simple observations.

Remark 2.2.28.

1. Every field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$ is necessarily infinite. This is because it must contain its prime field which is isomorphic to $\mathbb{Q}$, an infinite field. Therefore, finite fields need to have prime characteristic $p \neq 0$ by remark 2.2.15.

2. Since $\mathbb{Z}/p\mathbb{Z}$ is a prime field for every prime $p \in \mathbb{N}$, every finite field can be considered as a necessarily finite-dimensional $\mathbb{Z}/p\mathbb{Z}$-vector space. If it is of dimension $n \in \mathbb{N}$ as a vector space over $\mathbb{Z}/p\mathbb{Z}$, it has $|\mathbb{Z}/p\mathbb{Z}|^n = p^n$ elements, i. e. it is of prime power order.

3. It can be proven that the group of units $\mathbb{F}^\times$ of a finite field $\mathbb{F}$ is cyclic and of order $|\mathbb{F}| - 1$. With Fermat's little theorem, we find that $a^{|\mathbb{F}|-1} = 1$ for all $a \in \mathbb{F}^\times$ or $a^{|\mathbb{F}|} = a$ for all $a \in \mathbb{F}$.

Up to now, possibly multiple non-isomorphic fields with p^n elements could exists or actually none. The following theorem answers this question.

Theorem 2.2.29 (Classification of Finite Fields)**.** For any prime $p \in \mathbb{N}$ and every $n \in \mathbb{N}$ there exists (up to isomorphism) exactly one field with p^n elements, denoted $\mathbb{F}_{p^n}$. It is the splitting field of $X^{p^n} - X \in \mathbb{F}_p[X]$.

Now, we essentially know every finite field up to isomorphism. In the following, we want to use the simplification $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ for $p \in \mathbb{N}$ prime and define other finite fields as finite extension fields of this field.

Finite field extensions of a finite field, their Galois groups, and, thus, the automorphism group of any finite field exhibit remarkably simple structures which is shown in the following proposition.

Proposition 2.2.30. Let $\mathbb{L}/\mathbb{K}$ be a finite extension of a finite field $\mathbb{K}$ with characteristic $p \in \mathbb{N}$.

1. The extension $\mathbb{L}/\mathbb{K}$ is a Galois extension and simple. A primitive element is given by a generator of $\mathbb{L}^\times$, i. e. we have $\mathbb{L} = \mathbb{K}(a)$ for $\mathbb{L}^\times = \langle a \rangle$.

2. The Galois group $\mathrm{Gal}(\mathbb{L}/\mathbb{K})$ is cyclic and of order $[\mathbb{L} : \mathbb{K}]$. It is generated by the $\mathbb{K}$-automorphism $\phi\colon \mathbb{L} \to \mathbb{L}, x \mapsto x^{|\mathbb{K}|}$.

3. The group of automorphisms $\mathrm{Aut}(\mathbb{L})$ of $\mathbb{L}$ is given by $\mathrm{Aut}(\mathbb{L}) = \mathrm{Gal}(\mathbb{L}/\mathbb{F}_p) = \langle \phi_p \rangle$, i. e. generated by the Frobenius-automorphism $\phi_p\colon x \mapsto x^p$ as defined in remark 2.2.6.

Let us have a look at an explicit example. We want to combine all of what we have learnt about finite fields in the following example.

Example 2.2.31. At first, we want to find a generator of the cyclic group of units of the prime field $\mathbb{F}_7$ by first inspecting the multiplication table of $\mathbb{F}_7^\times$. We drop the label of equivalence classes for simplicity.

$\cdot$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

.

We see that every row and every column is a permutation of the elements of $\mathbb{F}_7 = \{0, 1, 2, 3, 4, 5, 6\}$ which is the multiplication with a fixed element given by the row or column. This is expected by the considerations of group actions in remark 2.1.21 and example 2.1.22. We can also read off the squares in $\mathbb{F}_7^\times$ which are the diagonal elements of the multiplication table. These are $1, 2, 4 \in \mathbb{F}_7$. They also form a group, the group of squares, usually denoted $(\mathbb{F}_7^\times)^2$.

To find a generating element of $\mathbb{F}_7^\times$, we need to consider powers of the elements to find their order. Since the order of the group $\mathbb{F}_7^\times$ is 6, we need to find an element $g \in \mathbb{F}_7^\times$ with $o(g) = 6$. Take for example $g = 3 \in \mathbb{F}_7^\times$:

$$3^1 = 3, 3^2 = 2, 3^3 = 3 \cdot 2 = 6, 3^4 = 3 \cdot 6 = 4, 3^5 = 3 \cdot 4 = 5 \text{ and } 3^6 = 3 \cdot 5 = 1$$

which can be read off of the multiplication table. This means that $3 \in \mathbb{F}_7^\times$ is a generator of $\mathbb{F}_7^\times$ and, thus, $\mathbb{F}_7^\times$ is cyclic which it should be.

Let us now consider a quadratic field extension of $\mathbb{F}_7$ given by the quotient $\mathbb{F}_7[X]/(X^2 - 6)$. From the multiplication table it can be seen that $6 = -1 \in \mathbb{F}_7$ is a non-square. Therefore, $X^2 - 6 \in \mathbb{F}_7[X]$ has no roots in $\mathbb{F}_7$ and is thus irreducible since it is of quadratic order. The field extension is also quadratic and, hence, leads to the field $\mathbb{F}_{7^2}$. The extension field is given by $\mathbb{F}_{7^2} = \mathbb{F}_7(\alpha)$ with $\alpha^2 = 6$, i. e. any element of $\mathbb{F}_{7^2}$ can be written in the form $a + b\alpha$ with $a, b \in \mathbb{F}_7$.

The action of the Frobenius map $\phi_7\colon x \mapsto x^7$ on $\mathbb{F}_{7^2}$ is given by its action on α since it is an $\mathbb{F}_7$-homomorphism. We find that $\phi_7(\alpha) = \alpha^7 = (\alpha^2)^3\alpha = 6^3\alpha = 6\alpha = -\alpha$. Therefore, $\phi_7(a + b\alpha) = a - b\alpha$. Hence, $\phi_7^2 = \mathrm{id}_{\mathbb{F}_{7^2}}$. The group of automorphisms $\mathrm{Aut}(\mathbb{F}_{7^2})$ is then given by $\mathrm{Aut}(\mathbb{F}_{7^2}) = \langle \phi_7 \rangle = \{\mathrm{id}_{\mathbb{F}_{7^2}}, \phi_7\}$.

As we have seen in the beginning of the example, squares in the group of units of a finite field form a subgroup which is of half the order of the group of units. Later, in the context of quadrics, we will consider polynomials in multiple variables with only quadratic terms and the number of their roots.

We will see that this will be dependent on whether some elements are squares in the finite field we want to consider or not. Therefore, we want to introduce a symbol, called *Legendre symbol* or, generalized, *Jacobi symbol*, which tells us whether a given element is a square in this finite field or not. Note that in the following we only want to consider finite fields of odd order because in even order some of the theorems and definition break down since the order of the group of units is not even any more.

Definition 2.2.32. Let $\mathbb{F}_q$ be a finite field with $q = p^n$ with $p \in \mathbb{N}$ odd and prime, and $n \in \mathbb{N}$.

1. The **group of squares** $\left(\mathbb{F}_q^\times\right)^2$ of $\mathbb{F}_q$ is given by $\left(\mathbb{F}_q^\times\right)^2 = \{x^2 \mid x \in \mathbb{F}_q^\times\} \subseteq \mathbb{F}_q^\times$. Its order is $|\left(\mathbb{F}_q^\times\right)^2| = \frac{1}{2}|\mathbb{F}_q^\times|$.

2. For $\mathbb{F}_q = \mathbb{F}_p$ we define the **Legendre symbol** $\left(\frac{a}{p}\right)$ of $a \in \mathbb{F}_p$ as

$$\left(\frac{a}{p}\right) := \begin{cases} 1 & \text{if a is a square in } \mathbb{F}_p \text{ and } a \neq 0 \in \mathbb{F}_p, \\ -1 & \text{if a is a non-square in } \mathbb{F}_p, \text{ and} \\ 0 & \text{if } a = 0 \in \mathbb{F}_p. \end{cases}$$

3. As a generalization, we define for $b \in \mathbb{F}_q$ the **Jacobi Symbol** as $\left(\frac{b}{q}\right) = \left(\frac{b}{p}\right)^r$, where on the right hand side the above defined Legendre symbol is used.

Remark 2.2.33.

1. $0 = 0^2 \in \mathbb{F}_q$ can also be considered a square but it is left out of the group of squares to actually form a group.

2. It can be proven that the Legendre symbol for fixed prime p is multiplicative, i. e. for $a, b \in \mathbb{F}_p$ we have $\left(\frac{a}{p}\right)\left(\frac{b}{p}\right) = \left(\frac{ab}{p}\right)$ which means that it forms a group homomorphism $\mathbb{F}_p^\times \to C_2$ from the group of units $\mathbb{F}_p^\times$ of a finite field $\mathbb{F}_p$ to the multiplicative cyclic group $C_2 = \{1, -1\}$ of order 2 with kernel $\left(\mathbb{F}_p^\times\right)^2$.

3. Note that an odd prime p is either $p \equiv 1 \pmod 4$ or $p \equiv 3 \pmod 4$. It can be shown by an easy calculation that only in the first case we find that $-1 \in \mathbb{F}_p$ is a square. In the other case, with $-1 \in \mathbb{F}_p$ being a non-square, it can be shown by the use of the Legendre symbol that either a or $-a$ is a square in $\mathbb{F}_p$.

We will later use these finite fields to describe event-based ontology with sequences of finite length embedded in a projective space of such a finite field.

2.3 Projective Spaces

Physics, in particular general relativity or modern quantum field theory, is often not only about numbers and differential equations, but also about geometry and its role in nature. Especially in general relativity, we find that the geometry of a spacetime is at the heart of the theory. Thus, we want to define some geometrical notions which will be the starting point of our description of spacetime. This will be slightly different to the usual approach using manifolds or vector spaces. In contrast, we want to use a more 'symmetric' geometry in which, in two dimensions, *lines* and *points* and their roles are interchangeable which will be made more precise in the following. These spaces are known as *projective spaces* and can be defined in two different but connected ways. They are rather homogeneous and lead to remarkable results, especially when considering their homomorphisms, so-called *projectivites*.

The first approach uses a more axiomatic or synthetic definition from incidence geometry. Here, a projective space is considered to consist of points and lines and a relation between them which could be spelled out in words as: "point p lies on line l." Points and lines need to satisfy certain conditions which are rather symmetric in their usage of points and lines. This approach leads to a more geometrical understanding of those spaces which will be useful to understand why we chose projective spaces as a description of spacetime, but it is rather laborious to calculate using this construction. It is in fact very useful in proving more theorems on the geometrical side of these spaces.

The second approach is more algebraic and, thus, more useful for calculations since there is an easy way to use coordinates, so-called *homogeneous coordinates*. Here, we start with a vector space over some field $\mathbb{K}$, usually a finite prime field $\mathbb{F}_p$, and consider as points of the projective space all one-dimensional subspaces of this vector space. This can also be done via an equivalence relation on the vector space and using the quotient space with respect to this equivalence relation.

The axiomatic way of defining a projective space can be connected to the algebraic definition by the construction of a coordinate field of this projective space which is possible if certain properties of the projective space are fulfilled. In the following we only want to consider those cases and ignore 'freaks' in the construction. As general reference we refer to [2] and [3]. In the following, many theorems and definitions are due to Veblen and Young. Thus, we refer to [16, 17] for proofs and as an additional general reference.

2.3.1 Definition from Incidence Geometry

At first, we want to start with the axiomatic or synthetic definition of a projective plane which is a two-dimensional projective space. This is done using incidence geometry which is the formal way of defining a geometry using the quote above "point p lies on line l". It clarifies the notions of "point", "line" and "lies on" which are the three ingredients of one of the following definition. At first, let us introduce the terminology *geometry*.

Definition 2.3.1.

1. A **geometry** $G = (X, I)$ is a tuple consisting of a set Ω and a subset $I \subseteq X \times X$, called **incidence relation**, satisfying the following conditions:
   a) The relation I is symmetric, i. e. $\forall (x, y) \in I\colon (y, x) \in I$.
   b) I is reflexive, i. e. $\forall x \in X\colon (x, x) \in I$.

   If $(x, y) \in I$, we say that x and y are **incident**.

2. A **flag** F of a geometry (X, I) is defined as a subset $F \subseteq X$ such that $F \times F \subseteq I$, i. e. all pairs (x, y) of elements of $x, y \in F$ are incident.
   It is called **maximal** if no more elements can be included, i. e. $\forall x \in X \setminus F$ we have that $F \cup \{x\}$ is not a flag.

3. The **rank** r of a geometry (X, I) is a natural number $r \in \mathbb{N}$ such that there exists a decomposition $X = X_1 \cup X_2 \cup \cdots \cup X_r$ with disjoint subsets $X_i \subseteq X, i = 1, \ldots, r$, such that each maximal flag contains exactly one element of each X_i. An element of X_i is called an **element of type** i.

Remark 2.3.2.

1. Note that this definition of incidence does not mean intersection of for example two lines. This will be defined later.

2. A rather natural incident relation I is given by the set-theoretic inclusion $\subseteq$ which may be turned symmetric by $(a, b) \in I$ if and only if $a \subseteq b$ or $b \subseteq a$.

Example 2.3.3.

1. The three-dimensional vector space $\mathbb{R}^3$ is a geometry with X being the set of all points, lines and planes in $\mathbb{R}^3$ and incidence given by symmetric inclusion.

2. For a point $p \in \mathbb{R}^3$, a line $l \subseteq \mathbb{R}^3$ with $p \in l$ and a plane $F \subseteq \mathbb{R}^3$ with $l \subseteq F$ we find that the following subsets of $\{p, l, F\}$ are flags:
   $\{p\}, \{l\}, \{F\}, \{p, l\}, \{p, F\}, \{l, F\}, \{p, l, F\}$.

3. A maximal flag in (X, I) as above is for example $\{p, l, F\}$. It can be shown that each maximal flag consists of a point, a line containing that point and a plane containing that line. Thus, the rank of this geometry is 3.

These definitions regarding geometries are rather rudimentary but are quite universal since they are so simple. A special case of a geometry is an *incidence structure* where points and lines are the defining subsets. This will be the starting point of the definition of a projective plane.

Definition 2.3.4.

1. An **incidence structure** (P, B, I) is a geometry (X, I) which is of rank 2. Here, $P = X_1 \subseteq X$ is the set of elements of type 1, called **points**, and $B = X_2 \subseteq X$ is the set of elements of type 2, called **blocks**.

2. The blocks $b \in B$ of an incidence structure (P, B, I) are called **lines** if for each $(p, q) \in P \times P$ with $p \neq q$ there exists exactly one block $l \in B$ such that $(p, l) \in I$ and $(q, l) \in I$.

3. An incidence structure where all blocks are lines is called a **linear space** and we usually write L instead of B for the set of lines.

4. For a given block $b \in B$ of an incidence structure (P, B, I), we call $\hat{b} = \{p \in P \mid (p, b) \in I\}$ a **pencil of points**.

5. Similarly, for a given point $p \in P$ of an incidence structure (P, B, I), we call $\hat{p} = \{b \in B \mid (p, b) \in I\}$ a **pencil of blocks**.

Remark 2.3.5.

1. As an alternative definition of an incidence structure (P, B, I), one can start with two building blocks, called points and lines, and define an incidence relation by the usual pairing of a point lying on a line, i. e. with the relation $\in$.

2. With the definition of a pencil, we can re-define our incidence relation for a point $p \in P$ and a block $b \in B$ by $(p, b) \in I$ if and only if $p \in \hat{b}$. This means that we can replace the set of blocks B by the set of pencils of points $\hat{B}$ and use the relation $\in$ instead of I. This can be thought of as separating the set of all points into smaller chunks and asking whether a given point is in one of these subsets corresponding to a block or line. This means that the set of lines can be regarded as a set of subsets of P.
   In the following, we will, for simplicity, not distinguish between these two formulations, in particular in the case of linear spaces.

To sum up some notations regarding linear spaces we give the following definition.

Definition 2.3.6. Let $(P, L, \in)$ be a linear space.

1. A point $p \in P$ is called **intersection** or **meet** of two lines $l_1, l_2 \in L$ if $p \in l_1$ and $p \in l_2$, denoted $p = l_1 \wedge l_2$.

2. A line $l \in L$ is called **connection** or **join** of two points $p_1, p_2 \in P$ if $p_1 \in l$ and $p_2 \in l$, denoted $l = p_1 \vee p_2$.

3. Two lines $l_1, l_2 \in L$ are called **parallel**, denoted $l_1 \mid\mid l_2$, if either $l_1 = l_2$ or $l_1 \cap l_2 = \emptyset$.

4. Points $p_1, p_2, \ldots \in P$ are called **collinear** if there exists a line $l \in L$ with $p_1, p_2, \ldots \in l$.

5. Lines $l_1, l_2, \ldots \in L$ are called **concurrent** if they intersect in one point, i. e. there is a point $p \in P$ such that $l_1 \cap l_2 \cap \ldots = \{p\}$.

6. Four points are called **quadrilateral** if no triple of them is collinear.

With all these notations, we are now ready to define a projective plane as an incidence structure, and in particular a linear space, which is probably the most symmetric in the sense that lines and points are interchangeable if changing the nomenclature of, for example, meet and join.

Definition 2.3.7. An incidence structure $(P, L, \in)$ is called a **projective plane**, denoted P^2, if it satisfies the following axioms:

(P1) For any two distinct points $p_1 \neq p_2 \in P$, there exists exactly one line $l \in L$ which is the join of both of them, i. e. $l = p_1 \wedge p_2$, so $p_1 \in l$ and $p_2 \in l$.

(P2) For any two distinct lines $l_1 \neq l_2 \in L$, there exists exactly one point $p \in P$ which is the meet of the two lines, i. e. $p = l_1 \vee l_2$, so $p \in l_1$ and $p \in l_2$.

(P3) There are four points quadrilateral, i. e. no more than two of them are collinear.

Remark 2.3.8. The last axiom of a projective plane to have four points quadrilateral is needed to disregard some degenerate cases.

Let us have a look at an example.

Example 2.3.9 (Fano Plane)**.** The most minimalistic example of a projective plane is the so-called **Fano plane**. It consists of seven points and seven lines connecting these points. A depiction can be found in Figure 2.1.

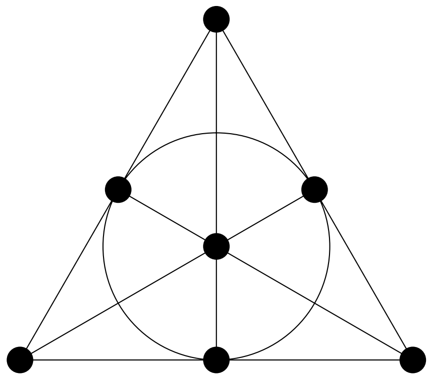


Figure 2.1: The Fano plane with its seven points and seven lines. The circle is to be understood as a line connecting the three points in the middle of the sides of the outer triangle.

One can immediately check that all three axioms of a projective plane are fulfilled since, for example, the three points at the corners of the outer triangle and the point at the centre of the circle are quadrilateral but are all pairwise connected.

We have seen that in a projective plane every two points are connected by a line and every two lines meet in one point. The name 'plane' suggests that this geometry is two-dimensional in the right definition. We now want to generalize this construction to what we will call *projective spaces*. These can be viewed as possibly higher-dimensional projective planes.

Definition 2.3.10. An incidence structure $(P, L, \in)$ is called a **projective space** if it satisfies the following axioms:

(PG1) Every line $l \in L$ contains at least three points.

(PG2) For any two distinct points $p_1, p_2 \in P$, there exists exactly one line $l \in L$ which is the join of p_1, p_2, i. e. $l = p_1 \vee p_2$ and $p_1, p_2 \in L$.

(PG3) If the lines $p_1 \vee p_2$ and $p_3 \vee p_4$ for points $p_1, \dots, p_4 \in P$ intersect, then the lines $p_1 \vee p_3$ and $p_2 \vee p_4$ also intersect.

(PG4) $|L| \geq 2$, i. e. there are at least two different lines.

Remark 2.3.11.

1. The third axiom (PG3) is also called the *Veblen-Young axiom*.
2. The first and the last axiom (PG1, PG4) are, again, needed to exclude some degenerate cases. The last one can be disregarded to also get a one-dimensional case with only one line.

Example 2.3.12. We will see in the next section that we can define a projective space for any field $\mathbb{K}$ and any dimension $n \in \mathbb{N}$. These are then denoted $\mathbb{K}\mathrm{P}^n$ or $\mathrm{PG}(n, \mathbb{K})$. They are constructed as the set of lines through the origin of the $(n+1)$-dimensional vector space $\mathbb{K}^{n+1}$.

Instead of using the axioms in the definition of a projective space for higher dimensional spaces, it is also possible to look at subspaces of rank 3 of a linear space and ask whether they are projective planes. This fact is stated in the following theorem by Veblen and Young.

Theorem 2.3.13 (Veblen, Young)**.** A linear space is a projective space if every subspace of rank 3 is a projective plane.

Given two incidence structures and, in particular, projective planes, we may ask whether these two can be transformed into each other by a re-labelling of the points and lines such that the incidence relation is respected and points are mapped to points and lines to lines. Of special interest are collinear points since they are the ingredients of a line. Bijective maps which map collinear points to collinear points will be called *collineations*. Using the same space as source and target of our mapping, we find the automorphisms of a projective space as the collineations of the space to itself. These are then those bijective maps from a projective space to itself which preserve the given incidence structure.

Definition 2.3.14. A bijective map $f\colon X = (P, L, \in) \to X' = (P', L', \in)$ between two projective spaces X, X' is called a **collineation** if it is a bijection $f\colon P \to P'$ on the set of points and a bijection $g\colon L \to L'$ on the set of lines which preserve the incidence relation, i. e. for a point $p \in l$ on a line $l \in L$ we find that $f(p) \in g(l)$. This also means that collinear points are mapped to collinear points. These maps are isomorphisms of projective spaces and form the **collineation group**.

For $X = X'$, the collineation group is the **automorphism group** of the projective space X.

To classify the projective spaces, it proves to be useful to define a notion of (*geometric*) *dimension*. This is done by defining *subspaces* of a projective space and looking for ascending chains of subspaces up to the full space. With this definition, we can define points, lines, planes and hyperplanes by their dimension and later classify all projective spaces.

Definition 2.3.15. Let $S = (P, L, \in)$ be a projective space.

1. A **subspace** of the projective space S is a subset $X \subseteq P$ such that any line which contains two points of X is again completely contained in X.

2. The **(geometric) dimension** of the space S is the largest natural number $n \in \mathbb{N}$ such that there exists an ascending chain of subspaces $\emptyset =: X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n = P$.

3. **Points** are defined as subspaces of dimension 0, **lines** are subspaces of dimension 1 and those of dimension 2 are called **planes**. If $n \in \mathbb{N}$ is the dimension of S, then a subspace of dimensions $n-1$ is called a **hyperplane**.

Remark 2.3.16. The notion of subspaces of a projective space can be directly generalized to the case of linear spaces as a subset of points together with lines consisting of these points such that each line with two points of this subset is contained completely in the subspace. The incidence relation is then given by a restriction to these subsets.

It is convenient to introduce coordinates in these geometries to work with these rather abstract spaces. This can be done as described in the next section if some extra properties are satisfied. There are several geometrical theorems which can be thought of as extra axioms to avoid some 'freaky' examples of geometries where this does not work.

We want to present the theorem by Desargues as one of these theorems since it holds in projective spaces of dimension one or at least three but produces interesting and non-intuitive examples in two dimensions if it does not hold. These are known as *Non-Desarguesian geometries*.

It can be shown that this property of a projective space to satisfy Desargues's theorem is needed to introduce coordinates over a field or, at least, a skew field. Therefore, we need to postulate this property separately to construct coordinates in all dimensions which will be done in the next section.

Theorem 2.3.17 (Desargues)**.** Let $(p_1, q_1), (p_2, q_2)$ and (p_3, q_3) be pairs of points in a projective space such that each pair (p_i, q_i) is collinear on a different line, respectively. The lines $p_1 \vee p_2$ and $q_1 \vee q_2$ meet in a point S_1, the lines $p_1 \vee p_3$ and $q_1 \vee q_3$ in a point S_2 and the lines $p_2 \vee p_3$ and $q_2 \vee q_3$ in a point S_3.

Then, the three points S_1, S_2, S_3 are collinear, i. e. lie on a common line l, the **axis of perspectivity**, if and only if the three lines $p_i \vee q_i, i \in \{1, 2, 3\}$, are concurrent at a point c, the centre of perspectivity.

As mentioned earlier, projective spaces of dimension two are curious cases where the theorem of Desargues need not hold. In dimension at least three, the following theorem by Veblen and Young states that this does not happen.

Theorem 2.3.18 (Veblen & Young)**.** In a projective space of dimension at least three the theorem of Desargues holds.

With the theorem of Desargues as an additional axiom for projective spaces of dimension two, we can now fully classify the projective spaces of given dimension up to isomorphism which is to be understood as a bijective map preserving the incidence structure of a projective space.

Theorem 2.3.19. Let S be a projective space of dimension $n \in \mathbb{N}$ satisfying the theorem of Desargues.

Then, S is isomorphic to a projective space $\mathbb{K}\mathrm{P}^n$ constructed over some (possibly skew) field $\mathbb{K}$ as the set of lines through the origin of $\mathbb{K}^{n+1}$.

Remark 2.3.20.

1. Note that a projective space of dimensions 0 or 1 is possible if we drop the last axiom (PG4). These are then just a single point with no lines or a single line, respectively.

2. A projective space of dimension 2 is equivalent to a projective plane, but only for the Desarguesian planes satisfying the theorem of Desargues it is possible to construct coordinates over some skew field.

3. In the case of a finite projective space, this can be done using a finite field since alls finite skew fields are actually fields.

Since we are in particular interested in finite projective spaces which satisfy the theorem of Desargues, we find that every finite projective space is isomorphic to $\mathbb{F}_q\mathrm{P}^n$ over some finite field $\mathbb{F}_q$ for some prime power $q = p^m$, prime $p \in \mathbb{N}$, $m \in \mathbb{N}$, and dimension $n \in \mathbb{N}$. Since these spaces are finite, we can count the number of points on a line and the number of points in general in such a projective space.

Proposition 2.3.21. Let S be a finite projective space of dimension $n \in \mathbb{N}$ which satisfies the theorem of Desargues.

Then, every line contains the same number of points $t + 1$. t is then said to be the **order** of this projective space. This number coincides with the order of the finite field over which the space can be coordinatized, i. e S is isomorphic to some $\mathbb{F}_q\mathrm{P}^n$ with a prime power $q = t$. The whole space has $q^n + q^{n-1} + \cdots + q + 1$ points.

In particular, a finite projective plane of order q has $q^2 + q + 1$ points, $q^2 + q + 1$ lines, every line contains $q + 1$ points and through every point there are $q + 1$ lines.

2.3.2 Algebraic Definition

We have seen that by requiring a projective space to satisfy the theorem of Desargues it is isomorphic to a projective space constructed over some (skew) field. Since we are in particular interested in finite projective spaces, we can reduce this construction to spaces constructed over some field. This construction and its implications to coordinates still need to be defined. We will see that every vector space over some field allows a *projectivization* to become a projective space. Since finite-dimensional vector spaces are all isomorphic to $\mathbb{K}^n$ for their base field $\mathbb{K}$ and dimension $n \in \mathbb{N}$, we will, in particular, work with the 'standard' projective space $\mathbb{K}\mathrm{P}^n$ over some field $\mathbb{K}$ and of dimension n which is the projectivization of the $\mathbb{K}$-vector space $\mathbb{K}^{n+1}$. In these, it is convenient to introduce so-called *homogeneous coordinates* which respect the homogeneous nature of the projective space.

At first, we start by defining an equivalence relation on a $\mathbb{K}$-vector space V without the zero vector by identifying vectors which only differ by multiplication of a non-zero scalar. The projective space $\mathrm{P}(V)$ formed from V is then the set of equivalence classes of $V \setminus \{0\}$ with respect to this equivalence relation.

Definition 2.3.22. Let V be a vector space over the field $\mathbb{K}$.

1. We define an equivalence relation $\sim$ on $V \setminus \{0\}$ by

$$v \sim w \iff \exists\, \lambda \in \mathbb{K}^\times : w = \lambda v$$

   for $v, w \in V \setminus \{0\}$.

2. The **projectivization** $\mathrm{P}(V)$ of V which makes V a projective space is defined as the set of equivalence classes of $V \setminus \{0\}$ under the equivalence relation $\sim$, i. e. $\mathrm{P}(V) := (V \setminus \{0\})/\sim\; = \{[v]_\sim \mid v \in V \setminus \{0\}\}$.

3. The map $\pi\colon V \to \mathrm{P}(V), v \mapsto [v]_\sim$ is the **canonical projection** onto the set of equivalence classes.

4. The projectivization of the vector space $V = \mathbb{K}^{d+1}$ for $d \in \mathbb{N}$ is denoted $\mathrm{PG}(d, \mathbb{K}) = \mathbb{K}\mathrm{P}^d = \mathrm{P}(\mathbb{K}^{d+1})$. Because finite-dimensional vector spaces of dimension $d+1$ over some field $\mathbb{K}$ are isomorphic to $\mathbb{K}^{d+1}$, we speak of $\mathbb{K}\mathrm{P}^d$ as the ('standard') **projective space of dimension** d **over** $\mathbb{K}$.

Remark 2.3.23.

1. Note that the dimension of a projective space differs from the dimension of its underlying vector space by 1 because of the identification of non-zero multiples of a vector.

2. The lines of a projectivization to become a projective space are given by the projectivization of planes in the corresponding vector space. In particular, the line $p \vee q$ for $p, q \in \mathbb{K}\mathrm{P}^n$ is given by the set $\{\lambda v + \mu w \mid (\lambda, \mu) \in \mathbb{K}^2 \setminus \{(0, 0)\}\}$ with $v, w \in \mathbb{K}^{n+1}$ representatives of the equivalence classes of p and q, respectively.

3. The number of elements of $\mathbb{F}_q\mathrm{P}^n$ for a finite field $\mathbb{F}_q$ can easily be calculated. It is given by $|\mathbb{F}_q\mathrm{P}^n| = \frac{|\mathbb{F}_q^{n+1}|}{|\mathbb{F}_q^\times|} = \frac{q^{n+1}}{q-1} = q^n + q^{n-1} + \ldots + q + 1$.

This definition via the equivalence relation exhibits the notion that in a projective space non-zero multiples of an element get identified. There is another, equivalent definition of a projective space in the algebraic way which is done by lines or rays through the origin and identifying all points on a ray.

Definition 2.3.24. Let V be a vector space over the field $\mathbb{K}$.

The **associated projective space** $\mathrm{P}(V)$ of V is defined as the set of rays of V, i. e. $\mathrm{P}(V) = \{\langle v\rangle \mid v \in V \setminus \{0\}\}$ where $\langle v\rangle$ denotes the ray generated by $v \in V$, i. e. $\langle v\rangle = \{\lambda v \mid \lambda \in \mathbb{K}\}$.

For $V = \mathbb{K}^{n+1}$, we write $\mathrm{P}(\mathbb{K}^{n+1}) = \mathbb{K}\mathrm{P}^n$ for the standard n-dimensional projective space.

Remark 2.3.25. There is also the notion of a *dual projective space* $\mathrm{P}(V)^*$ of the associated projective space $\mathrm{P}(V)$ of a (finite-dimensional) vector space V over some field $\mathbb{K}$ as the associated projective space of the dual vector space $V^* = \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$ of $\mathbb{K}$-linear maps from V to $\mathbb{K}$, i. e. $\mathrm{P}(V)^* = \mathrm{P}(V^*)$. This can be regarded as the space of hyperplanes through the origin.

In two projective dimensions, the idea should be like this: In three-dimensional 'normal' space, we start at looking at the intersection of a plane not through the origin with lines through the origin. These points are then part of the projective space. Moving the plane on a line along its normal direction through the origin will then lead to the same points in projective space since they only differ by a non-zero scalar even though the intersection points are not identical any more. Additionally, there are lines which are parallel to the chosen plane. These will become so-called *points at infinity* which may be viewed as additional points in comparison to a vector space of two dimensions for becoming a projective space.

If the (finite-dimensional) vector space V is given coordinates which is essentially a choice of isomorphism between V and $\mathbb{K}^{\dim_{\mathbb{K}}(V)}$, then we can also find a notion of coordinates on the projective space $\mathrm{P}(V)$, so-called *homogeneous coordinates*. This representation intrinsically captures the nature of projective space and its points being invariant under non-zero scalar multiplication.

Definition 2.3.26. Let V be an $(n+1)$-dimensional vector space over the field $\mathbb{K}$. Let $(x_0, x_1, \ldots, x_n)^T \in \mathbb{K}^{n+1}$ be the coordinates of a vector $v \in V$.

The **homogeneous coordinates** of $[v]_\sim \in \mathrm{P}(V)$ are denoted $[x_0 : x_1 : \ldots : x_n]$ where the colons indicate the property that for every $\lambda \in \mathbb{K}^\times$ we have $[\lambda x_0 : \lambda x_1 : \ldots : \lambda x_n] = [x_0 : x_1 : \ldots : x_n]$.

The usage of homogeneous coordinates lets us default to the standard n-dimensional projective space $\mathbb{K}\mathrm{P}^n$ for a given field $\mathbb{K}$. In the following, we will mostly use $\mathbb{K}\mathrm{P}^n$ for simplification of calculations and in order to avoid the specification of a coordinate system of a more abstract vector space.

In the definition of homogeneous coordinates, we implicitly make the choice of a basis of the vector space V which is then used to define coordinates on V. An analogous construction is also known in projective geometry as a so-called *projective frame* or *projective basis*.

Definition 2.3.27. Let V be an $(n+1)$-dimensional vector space over the field $\mathbb{K}$ and $\mathrm{P}(V)$ its associated projective space.

A **projective frame** or **projective basis** of $\mathrm{P}(V)$ is an $(n+2)$-tuple of points of $\mathrm{P}(V)$ such that no hyperplane contains $n+1$ of them.

Remark 2.3.28. A projective frame of the associated projective space $\mathrm{P}(V)$ of an $(n+1)$-dimensional $\mathbb{K}$-vector space V can also be characterized by an $(n+2)$-tuple of points of $\mathrm{P}(V)$ such that $n+1$ points are the image of a basis of V under the canonical projection $\pi\colon V \to \mathrm{P}(V)$ and one is the image of the sum of the elements of the basis, which can be achieved by a rescaling.

Example 2.3.29. In the case of $\mathrm{P}(\mathbb{K}^{n+1}) = \mathbb{K}\mathrm{P}^n$, there is a canonical choice of basis, the canonical basis $(e_0, \ldots, e_n)$. Therefore, there is also the notion of a *canonical frame* of $\mathbb{K}\mathrm{P}^n$ which is the image of $(e_0, \ldots, e_n)$ under the canonical projection $\pi\colon \mathbb{K}^{n+1} \to \mathbb{K}\mathrm{P}^n$ and the image of the sum $e_0 + \ldots + e_n$, i. e. $(\pi(e_0), \ldots, \pi(e_n), \pi(e_0 + \ldots + e_n))$.

Since we have defined projective spaces from vector spaces, we are also interested in in the induced action of linear maps between vector spaces over the same field to the associated projective spaces. Note that the exclusion of the zero-ray in our definition of a projective space requires the linear map to be at least injective.

Additionally, as a sanity check, we find that linear maps respect the equivalence relation on the respective vector spaces since they respect scalar multiplication, i. e. for a linear map $f\colon V \to W$ for two $\mathbb{K}$-vector spaces V, W we have $f(\lambda v) = \lambda f(v)$ for any $\lambda \in \mathbb{K}$. Thus, the images of an equivalence class of vectors is, again, an equivalence class in the image of f. Hence, we may take $[v] \mapsto [f(v)]$ as the corresponding mapping of equivalence classes from $\mathrm{P}(V)$ to $\mathrm{P}(W)$ which is well-defined as discussed above.

Furthermore, we find that linear maps which only differ by a non-zero scalar, i. e. $g = \lambda f$ for $\lambda \in \mathbb{K}^\times$ and linear maps $f, g\colon V \to W$, act the same on the equivalence classes with respect to the equivalence relation used in the definition of the associated projective space. Additionally, it can be shown that two linear maps induce the same action on the associated projective spaces if and only if they differ by multiplication of a non-zero scalar. Thus, non-zero scalar multiples of linear maps should be identified.

If we now restrict to endomorphisms or, more precisely, automorphisms $\mathrm{Aut}(V) = \mathrm{GL}(V)$ of a vector space V, we can describe the induced mappings on the associated projective spaces by the concept of the *projective linear group* $\mathrm{PGL}(V)$ which is the induced action of the general linear group $\mathrm{GL}(V)$ of V on the projectivization $\mathrm{P}(V)$.

Definition 2.3.30. Let V be a vector space over the field $\mathbb{K}$, $\mathrm{GL}(V)$ its automorphism group and $\mathrm{P}(V)$ its associated projective space.

The **projective linear group** $\mathrm{PGL}(V)$ is defined as the quotient group of $\mathrm{GL}(V)$ by its centre $Z(\mathrm{GL}(V)) = \mathbb{K}^\times \mathrm{id}_V \subseteq \mathrm{GL}(V)$, i. e. $\mathrm{PGL}(V) = \mathrm{GL}(V)/Z(\mathrm{GL}(V))$. Its elements are called **projective linear transformations**, **homographies** or **projectivities**.

If $V = \mathbb{K}^n, n \in \mathbb{N}$, we write $\mathrm{PGL}(\mathbb{K}^n) = \mathrm{PGL}(n, \mathbb{K})$.

Remark 2.3.31.

1. Note that for $V = \mathbb{K}^n$ we have $\mathrm{PGL}(\mathbb{K}^n) = \mathrm{PGL}(n, \mathbb{K})$ but the associated projective space $\mathbb{K}\mathrm{P}^{n-1}$ is of projective or geometrical dimension $n-1$. This notion comes from the construction of $\mathrm{PGL}(n, \mathbb{K})$ as a quotient of $\mathrm{GL}(n, \mathbb{K})$ whereas the superscript in the notation $\mathbb{K}\mathrm{P}^{n-1}$ denotes the projective dimension of the projective space.

2. Since for finite-dimensional vector spaces linear maps can be represented by matrices, this can also be done in the case of the induced action on the corresponding projective spaces by identifying non-zero scalar multiples of matrices. The matrices then act by usual matrix-vector multiplication on the homogeneous coordinates.

3. Note that, as in the case of bases of a vector space, given two projective frames, there is exactly one (up to non-zero rescaling) homography which maps one to the other.

It is quite obvious that these projective linear transformations are collineations of $\mathrm{P}(V)$. The question might now be whether the projective linear group forms the whole collineation group of a projective space if it is Desarguesian since then a classification of projective spaces is easily possible as shown earlier. The answer is given by the *fundamental theorem of projective geometry.* We find that the projective linear group is in general a proper subgroup of the collineation group.

Theorem 2.3.32 (Fundamental Theorem of Projective Geometry)**.** Let $\mathbb{L}$ be a field with prime field $\mathbb{K}$. Let $P = \mathbb{L}\mathrm{P}^n$ with $n \geq 2$.

The collineation group of P is given by the **projective semilinear group** $\mathrm{P\Gamma L}(n+1, \mathbb{L}) = \mathrm{PGL}(n+1, \mathbb{L}) \rtimes \mathrm{Gal}(\mathbb{L}/\mathbb{K})$. The Galois group $\mathrm{Gal}(\mathbb{L}/\mathbb{K})$ of the field extension $\mathbb{L}/\mathbb{K}$ acts as an **automorphic collineation** on the homogeneous coordinates by acting on every coordinate individually, i. e. for $\sigma \in \mathrm{Gal}(\mathbb{L}/\mathbb{K})$ and $x = [x_0 : \ldots : x_n]$ we have $\sigma(x) \coloneqq [\sigma(x_0) : \ldots : \sigma(x_n)]$.

Remark 2.3.33.

1. Since all Desarguesian projective projective spaces are isomorphic to a projectivization of some vector space over a (skew) field, this also works as a classification of the automorphisms of general projective spaces of geometric dimension at least two.

2. The case of the projective line which is a projective space of geometric dimension one, is a little more subtle since all points are on one line. Thus, the collineation group is the symmetric group on the points of this line. Except for the cases of $\mathbb{K} = \mathbb{F}_2, \mathbb{F}_3$, the projective linear group is a proper subgroup of this symmetric group.

3. Note that in the case of $\mathbb{L}$ being a prime field we find that the semilinear and the linear projective group coincide if the projective dimension is at least two.

The twisting of the projective linear group by the Galois group of the field extension can be understood as a choice of a 'linear structure' in the projective space. Thus, a projective space exhibits a form of a semilinear structure which is preserved by the collineation group.

2.4 Affine Spaces

We have seen that a projective space can be constructed in multiple ways and is a rather homogeneous and symmetric object, but it is rather difficult to define a notion of translation in this setting since the usual vector addition breaks down under the identification with respect to the equivalence relation. In order to describe the motion of a particle in the physical spacetime we need to define velocities which can be regarded as translations vectors from one event to its successor. Thus, a useful setting for our physical spacetime is a so-called *affine space* where there exist translations between different points such that the composition of two translations is the translation with respect to the sum of the translation vectors. From a projective space, we can construct an affine space by the choice of a suitable *hyperplane at infinity* which can be thought of as a given parallel direction of translation in this homogeneous space. This construction also works in the reverse direction and will be, among others, presented in subsection 2.4.1.

Let us start by defining an affine space in the algebraic way. Note that there is also the notion of an affine plane form incidence geometry which will not be discussed for the sake of simplicity of calculations.

Definition 2.4.1.

1. An **affine space** $(P, V, +)$ over the field $\mathbb{K}$ is a triple consisting of a non-empty set of points $P \neq \emptyset$ together with a finite-dimensional $\mathbb{K}$-vector space V and a map $+\colon P \times V \to P, (p, v) \mapsto p + v$ such that the following is satisfied:
   - (A1) For all $p \in P, u, v \in V\colon (p + u) + v = p + (u + v)$ where the last $+$ is the addition in V.
   - (A2) For arbitrary $p \in P$ we have $p + v = p$ if and only if $v = 0 \in V$.
   - (A3) For all $p, q \in P$ there exists some $v \in V$ such that $q = p + v$.
2. The dimension of an affine space $(P, V, +)$ is the dimension of V as a $\mathbb{K}$-vector space.
3. The map $\cdot + v\colon P \to P, p \mapsto p + v$ for a fixed $v \in V$ is called **translation by** v and v is called the **translation vector** of this translation.

Remark 2.4.2. It can be easily shown that the vector $v \in V$ in axiom (A3) is unique for a fixed pair of points $p, q \in P$.

Example 2.4.3. The prime example of an affine space over the field $\mathbb{K}$ is $\mathbb{A}^n := (\mathbb{K}^n, \mathbb{K}^n, +)$ of the vector space $V = \mathbb{K}^n$ as the set of points and as translation vectors and usual vector addition as translation for any $n \in \mathbb{N}$. This can also be done with every other finite-dimensional vector space V over the field $\mathbb{K}$.

In order to classify all affine spaces up to isomorphism, we first need to define *affine maps* which are structure preserving maps of affine spaces. If the source and target space coincide and the map is bijective, we speak of *affinities*.

Definition 2.4.4. Let $(P, V, +)$ and $(P', V', +')$ be affine spaces over the field $\mathbb{K}$.

1. A map $f\colon P \to P'$ is called an **affine map** or **affine homomorphism** if there exists some linear map $\phi\colon V \to V'$ such that for all $p \in P$ and $v \in V$ we have $f(p + v) = f(p) +' \phi(v)$.

2. If an affine map $f\colon P \to P'$ is bijective, we speak of an **affine isomorphism**.

3. If $(P', V', +') = (P, V, +)$, an affine isomorphism $f\colon P \to P$ is called an **affinity** or **affine transformation**.

Remark 2.4.5.

1. Note that affinities are just the automorphisms of an affine space.

2. It can be shown that the linear map $\phi\colon V \to V'$ corresponding to the affine map $f\colon P \to P'$ is unique.

Example 2.4.6. The translations by vectors of the corresponding vector space of an affine space are automorphisms of this affine space since their inverses are given by the translation by the inverse vector.

The definition of affine isomorphisms allows us to classify all affine spaces over a given field by their dimension as it is done for vector spaces. The prototype of an n-dimensional affine space over a field $\mathbb{K}$ is $\mathbb{A}^n$. We can also classify the structure of affine maps $f\colon \mathbb{A}^n \to \mathbb{A}^m$ by the use of $(m \times n)$-matrices as follows.

Theorem 2.4.7.

1. Every affine space over the field $\mathbb{K}$ is isomorphic to $\mathbb{A}^n$ for some $n \in \mathbb{N}$, the dimension of the affine space.

2. Every affine map $f\colon \mathbb{A}^n \to \mathbb{A}^m$ for $n, m \in \mathbb{N}$ is given by $f(x) = Ax + b$ for some $A \in \mathrm{Mat}(m \times n, \mathbb{K})$ and $b \in \mathbb{K}^m$.

Remark 2.4.8.

1. The proof of the first point of this theorem works because of assumptions (A1) and (A3) in the definition of an affine space. These imply that the structure of the affine space is just the structure of its underlying vector space shifted by some choice of 'origin point' which is arbitrary but fixed, i. e. for some point p in the affine space as 'origin' we have that every other point $q = p + v$ for some vector v. If we have another point o and want to find the translation vector form q to o, we just need the translation vector from p to o, e. g. $o = p + w$, and find that $o = q + (w - v)$.

2. To write affine maps $f\colon \mathbb{A}^n \to \mathbb{A}^m, x \mapsto Ax + b$ as in the theorem more compactly, we can blow up the elements of $\mathbb{A}^n$ and $\mathbb{A}^m$ by an additional entry with a $1 \in \mathbb{K}$ in it, i. e. for $x \in \mathbb{A}^n$ we use $(x^T, 1)^T \in \mathbb{K}^{n+1}$ and analogously for $y \in \mathbb{A}^m$, and use an $((m+1) \times (n+1))$-matrix of the form $\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}$ such that

$$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} Ax + b \\ 1 \end{pmatrix} \in \mathbb{K}^{m+1}.$$

We will use these affine spaces to model the physical four-dimensional spacetime as a subspace of the projective space. Thus, these techniques will later be very useful.

2.4.1 Affine Spaces from Projective Spaces

An affine space can be obtained from a projective space by a construction called *slicing* which is done by the choice of some hyperplane of the projective space which will then be 'cut away' to get the affine space. Conversely, an affine space can be made a projective space by the addition of so-called *points at infinity*. This method is known as *projective completion*. This is done by the addition of points which are equivalence classes of parallel lines in the affine space. These techniques are useful to switch between those two concepts and get a better understanding of the structure of a projective space.

Firstly, we want to define the term *hyperplane* in a projective space in a more algebraic manner. We have already defined a hyperplane to be a subspace of a projective space of co-dimension 1, i. e. if the projective space is of (geometric) dimension $n \in \mathbb{N}$, a hyperplane is a subspace of (geometric) dimension $n - 1$. In the following, we want to consider the projective space $\mathbb{K}\mathrm{P}^n$ for some field $\mathbb{K}$ and $n \in \mathbb{N}$ for simplicity.

Definition 2.4.9. Let $\mathbb{K}$ be a field and $n \in \mathbb{N}$.

A **hyperplane** $H \subseteq \mathbb{K}\mathrm{P}^n$ of the projective space $\mathbb{K}\mathrm{P}^n$ is the subset $H = \{[p] \in \mathbb{K}\mathrm{P}^n \mid h^T p = 0\} \subseteq \mathbb{K}\mathrm{P}^n$ for some $h \in \mathbb{K}^{n+1}$ where $p \in \mathbb{K}^{n+1}$ is a pre-image of $[p] \in \mathbb{K}\mathrm{P}^n$ under the canonical projection. It is represented by its **normal vector** $h \in \mathbb{K}^{n+1}$ or, alternatively, by the homogeneous coordinates $[h] \in \mathbb{K}\mathrm{P}^n$ thereof.

Remark 2.4.10.

1. Note that the condition $h^T p = 0$ for the pre-image $p \in \mathbb{K}^{n+1}$ of some point $[p] \in \mathbb{KP}^n$ under the canonical projection is well-defined, since different pre-images only differ by a non-zero scalar, i. e. for $p' \in \mathbb{K}^{n+1}$ a pre-image of $[p] \in \mathbb{KP}^n$, we have that $p' = \lambda p$ for some $\lambda \in \mathbb{K}^\times$. Hence, $h^T p = 0 \iff h^T p' = h^T(\lambda p) = \lambda(h^T p) = 0$ with $\lambda \in \mathbb{K}^\times$.
   The same can be done to confirm that, indeed, only the homogeneous coordinates $[h] \in \mathbb{KP}^n$ of the normal vector $h \in \mathbb{K}^{n+1}$ of the hyperplane $H \subset \mathbb{KP}^n$ are necessary for h to represent the hyperplane H since non-zero multiples of h lead to the same hyperplane H.

2. The transformation behaviour of the normal vector $h \in \mathbb{K}^{n+1}$ of a hyperplane $H \subseteq \mathbb{KP}^n$ under some projectivity $f\colon \mathbb{KP}^n \to \mathbb{KP}^n$ represented by a matrix $G \in \mathrm{PGL}(n+1, \mathbb{K})$ can be deduced under the assumption that points on the hyperplane should be mapped onto points of the image of the hyperplane under f. This means for $p \in H$, i. e. $h^T p = 0$, and $\tilde{h} \in \mathbb{K}^{n+1}$ the normal vector of the image of the hyperplane that $\tilde{h}^T f(p) = \tilde{h}^T G p = 0$. For $\tilde{h} = (G^{-1})^T h$, this is immediately given since then $\tilde{h}^T G p = ((G^{-1})^T h)^T G p = h^T G^{-1} G p = h^T p = 0$. Hence, we find that the normal vector h of the hyperplane H transforms as $h \mapsto (G^{-1})^T h$ under the projectivity f represented by $G \in \mathrm{PGL}(n+1, \mathbb{K})$.

Example 2.4.11. Consider for any field $\mathbb{K}$ the normal vector $h = (1, 0, 0)^T \in \mathbb{K}^3$ of the hyperplane $H \subset \mathbb{KP}^2$. The homogeneous coordinates of a point $p \in \mathbb{KP}^2$ are given by $[x_0, x_1, x_2]$. Thus, for the point p to be in the hyperplane H it needs to have $x_0 = 0$, since $(1, 0, 0)(x_0, x_1, x_2)^T = x_0$. Therefore, the hyperplane H is given by points p of the projective space $\mathbb{KP}^2$ with homogeneous coordinates such that the first coordinate is 0, i. e. $H = \{[0 : x_1 : x_2]\} \subseteq \mathbb{KP}^2$.

With this well-defined definition of a hyperplane in a projective space, we can now make the following proposition which describes the structure of a projective space with respect to a given hyperplane.

Proposition 2.4.12. Let $\mathbb{K}$ be a field and $n \in \mathbb{N}$ and let $H \subseteq \mathbb{KP}^n$ be a hyperplane of the projective space $\mathbb{KP}^n$ with normal vector $h \in \mathbb{K}^{n+1}$.

1. The projective space $\mathbb{KP}^n$ is the union of two disjoint subsets $H, G \subseteq \mathbb{KP}^n$ with $G = \{[p] \in \mathbb{KP}^n \mid h^T p \neq 0\}$, i. e. $\mathbb{KP}^n = H \cup G$ with $H \cap G = \emptyset$.

2. The hyperplane H forms a projective space of dimension $n - 1$.

3. The set $G_1 = \{p \in [p] \in G \mid h^T p = 1\} \subseteq \mathbb{K}^{n+1}$ is an affine space of dimension n.

Proof. Let $[x_0 : \ldots : x_n]$ be homogeneous coordinates on $\mathbb{KP}^n$ such that w.l.o.g. $h = (0, \ldots, 0, 1)^T \in \mathbb{K}^{n+1}$ which can always be achieved by a rescaling and a coordinate transformation. Then, the hyperplane H takes the form $H = \{[x_0 : \ldots : x_{n-1} : 0] \in \mathbb{KP}^n\}$ and $G = \{[x_0 : \ldots : x_{n-1} : 1] \in \mathbb{KP}^n\}$.

1. For $p \in [p] \in \mathbb{K}\mathrm{P}^n$ we have either $h^T p = 0$ or $h^T p \neq 0$, but never both. Furthermore, as stated before, the condition $h^T p = 0$ and additionally $h^T p \neq 0$ are both well-defined in a projective space, i. e. they are independent of the choice of representative of an equivalence class. Thus, we have the decomposition $\mathbb{K}\mathrm{P}^n = H \cup G$ with $H \cap G = \emptyset$.

2. The set $\bar{H} := \{v \in \mathbb{K}^{n+1} \mid h^T v = 0\} \subseteq \mathbb{K}^{n+1}$ forms a linear subspace of $\mathbb{K}^{n+1}$ since $h^T(\lambda v + \mu w) = \lambda h^T v + \mu h^T w$ for any $\lambda, \mu \in \mathbb{K}$ and $v, w \in \mathbb{K}^{n+1}$ and, obviously, $0 \in \bar{H}$. This subspace is of dimension n since there is one linear condition imposed on the vectors. As remarked before, the condition $h^T p = 0$ is well-defined even in the projective sense. Thus, the projectivization of $\bar{H}$, $\mathrm{P}(\bar{H})$, is isomorphic to H and to $\mathbb{K}\mathrm{P}^{n-1}$ by the isomorphism $H \to \mathbb{K}\mathrm{P}^{n-1}, [x_0 : \ldots : x_{n-1} : 0] \mapsto [x_0 : \ldots : x_{n-1}]$. Therefore, we found $H \cong \mathbb{K}\mathrm{P}^{n-1}$.

3. With the convention above, we have $G_1 = \{(x_0 : \ldots : x_{n-1} : 1)^T \in \mathbb{K}^{n+1}\} = \{x \in \mathbb{K}^{n+1} \mid x_n = 1\}$. Every point of G_1 can be obtained from $(0, \ldots, 0, 1)^T \in \mathbb{K}^{n+1}$ by adding some vector $x \in \bar{H}$, i. e. $h^T x = 0$. $\bar{H}$ can be seen as the vector space $\mathbb{K}^n$ and, thus, we find the isomorphism $G_1 \to \mathbb{A}^n, (x_0, \ldots, x_{n-1}, 1)^T \mapsto (x_0, \ldots, x_{n-1})$. Hence, G_1 is an affine space of dimension n.

□

With consecutive applications of the methods shown in the proposition above to the hyperplane H by a choice of a hyperplane of H, we find the following corollary.

Corollary 2.4.13. The projective space $\mathbb{K}\mathrm{P}^n$ for a field $\mathbb{K}$ and $n \in \mathbb{N}$ allows a decomposition of the form $\mathbb{K}\mathrm{P}^n = \mathbb{K}^n \cup \mathbb{K}^{n-1} \cup \ldots \cup \mathbb{K}^1 \cup \mathbb{K}^0$ into affine spaces.

Remark 2.4.14. This consecutive procedure of choosing a hyperplane with respect to which an affine space is cut off is also referred to as **slicing** or **dehomogenization** since the projective space is sliced into different affine spaces along some hyperplanes and is, thus, no longer homogeneous.

Example 2.4.15. A rather standard way of arriving at this decomposition in, e. g. two dimensions can be done like this: Let us start with the projective space $\mathbb{K}\mathrm{P}^2$ and the hyperplane H with normal vector $(0, 0, 1)^T$. With that, we split the projective space into a set H of points with homogeneous coordinates where the last one is zero and into a set G of points where the last coordinates is non-zero, i. e. $H = \{[x_0 : x_1 : 0]\} \subseteq \mathbb{K}\mathrm{P}^2$ and $G = \{[x_0 : x_1 : x_2] \mid x_2 \neq 0\} = \{[y_0 : y_1 : 1]\} \subseteq \mathbb{K}\mathrm{P}^2$, i. e. an affine subspace.

Next, we split H into two disjoint subsets with respect to the hyperplane H_1 with normal vector $(0, 1, 0)^T$ in the same fashion, namely $H_1 = \{[x_0 : 0 : 0]\} = \{[1 : 0 : 0]\} \subseteq H$ and $G_1 = \{[x_0 : x_1 : 0] \mid x_1 \neq 0\} = \{[z_0 : 1 : 0]\} \subseteq H$.

The next step would be the splitting of H_1 into two disjoint subsets with respect to the normal vector $(1, 0, 0)^T$, but we find that this results in just $G_2 = H_1 \subseteq H_1$ and $H_2 = \emptyset \subseteq H_1$. Therefore, we find the decomposition $\mathbb{K}\mathrm{P}^2 = G \cup G_1 \cup G_2$, i. e. $\mathbb{K}\mathrm{P}^2 = \{[y_0 : y_1 : 1]\} \cup \{[z_0 : 1 : 0]\} \cup \{[1 : 0 : 0]\}$, whereby the point $[1 : 0 : 0]$ is also referred to as **point at infinity** and G_1 as **line at infinity**.

This procedure can be viewed as a covering of the projective space by *affine charts* and only considering the newly formed hyperplane. A quite natural choice for such an affine chart, which is basically just a choice of an affine space inside of the projective space, is setting one of the homogeneous coordinates to 1, e. g. the last one. With every affine chart, we get a splitting of the projective space into an affine space and a hyperplane, called the *hyperplane at infinity* relative to the affine space.

Definition 2.4.16. Let $\mathbb{K}$ be a field and $n \in \mathbb{N}$.

1. An **affine chart** of the projective space $\mathbb{K}\mathrm{P}^n$ is an isomorphism $\mathbb{A}^n \to G \subseteq \mathbb{K}\mathrm{P}^n$. The complement $H = \mathbb{K}\mathrm{P}^n \setminus G$ is a hyperplane, called the **hyperplane at infinity** relative to $G \cong \mathbb{A}^n$. Its points are referred to as **points at infinity**.

2. The i**th standard affine chart** is the map $(x_0, \ldots, x_{n-1})^T \mapsto [x_0 : \ldots : x_{i-1} : 1 : x_{i+1} : \ldots : x_n]$ for $i \in \{0, \ldots, n\}$ given homogeneous coordinates $[y_0 : \ldots : y_n] \in \mathbb{K}\mathrm{P}^n$.

3. For $x = [x_0 : \cdots : x_n] \in \mathbb{K}\mathrm{P}^n$ with $x_i \neq 0, i = 0, \ldots, n$, $(y_0, \ldots, y_{i-1}, 1, y_{i+1}, \ldots, y_n)^T$ with $y_j = \frac{x_j}{x_i}$ are called **affine coordinates** of x corresponding to the ith standard affine chart.

Remark 2.4.17.

1. Note that every affine chart gives rise to a hyperplane at infinity relative to the affine space, and, hence, only the use of all $n + 1$ standard affine charts on $\mathbb{K}\mathrm{P}^n$ covers the whole projective space.

2. In the following, we will mostly use the nth standard affine chart to describe our physical spacetime as an affine space in a projective space, i. e. $\mathbb{A}^n \cong G_n := \{[x_0 : \ldots : x_{n-1} : 1] \in \mathbb{K}\mathrm{P}^n\}$. Accordingly, we will refer to $H_\infty = \{x \in \mathbb{K}\mathrm{P}^n \mid x_n = 0\}$ as hyperplane at infinity.

3. Affine charts also give rise to the opposite procedure of turning an affine space into a projective space, known as **projective completion**. This is done by also adding points with homogeneous coordinates of the form $[x_0 : \ldots : x_{i-1} : 0 : x_{i+1} : \ldots : x_n]$ which can be seen as representatives of equivalence classes of parallel lines in the affine space. The latter can also be done in the case of incidence geometry.

Up to now, we have only studied the objects affine spaces as a subset of the objects projective spaces. In complete analogy, the constructions above can be used to describe affine transformations as projective transformations given an affine chart, and discuss the action of a projectivity on a chosen affine subspace in the projective space.

Since an affine space can be considered a subspace of some projective space as the image of some affine chart, we see that the action of a projectivity on this affine space is just its action on the projective space restricted to the subspace of the affine space.

Example 2.4.18. To make this more concrete, we may look at the projective space $\mathbb{K}\mathrm{P}^n$ of dimension $n \in \mathbb{N}$ for some field $\mathbb{K}$ and consider as image of the affine chart the subset $G = \{[y_0 : \ldots : y_{n-1} : 1]\} \subseteq \mathbb{K}\mathrm{P}^n$. A projectivity $f\colon \mathbb{K}\mathrm{P}^n \to \mathbb{K}\mathrm{P}^n$ can be represented by an $((n+1)\times(n+1))$-matrix $A \in \mathrm{GL}(n+1, \mathbb{K})$ where, as usual, non-zero scalar multiples are identified. We can write the matrix A in a more suitable way as

$$A = \begin{pmatrix} B & t \\ h^T & c \end{pmatrix}$$

with $B \in \mathrm{Mat}(n \times n, \mathbb{K}), t, h \in \mathbb{K}^n$ and $c \in \mathbb{K}$. We find that the action of f on an element $x = [y^T : 1] \in G$ of the affine subspace G is then given by

$$Ax = \begin{pmatrix} B & t \\ h^T & c \end{pmatrix} \begin{pmatrix} y \\ 1 \end{pmatrix} = \begin{pmatrix} By + t \\ h^T y + c \end{pmatrix} = \begin{pmatrix} (h^T y + c)^{-1}(By + t) \\ 1 \end{pmatrix}$$

where in the last step we assumed that $h^T y + c \neq 0$ and identified non-zero scalar multiples. Otherwise, this point would be mapped to a point at infinity. This is the case if x is an element of the hyperplane with normal vector $(h^T, c)^T \in \mathbb{K}^{n+1}$.

The action of the projectivity f onto the affine subspace G can now be interpreted as an action of f onto the affine space $\mathbb{A}^n$ by the identification $[y^T : 1] \mapsto y$. Thus, it is given by $y \mapsto (h^T y + c)^{-1}(By + t)$ if $h^T y + c \neq 0$. This is in general not an affine transformation because of the pre-factor $(h^T y + c)^{-1}$ which is, in general, non-affine with respect to y.This problem vanishes if h vanishes, i. e. $h = 0$, making the induced action affine.

Remark 2.4.19. A projectivity with $h \neq 0$ can be thought of as tilting the hyperplane at infinity since points of the affine space relative to this hyperplane can be mapped outside of the affine space or, conversely, points at infinity may be mapped to points in the affine space if its definition is not changed accordingly.

The procedure above can be generalized for general affine charts and projectivities if they are compatible, meaning that the projectivity should map the affine subspace onto itself.

Proposition 2.4.20. Let $\mathbb{K}$ be a field and $n \in \mathbb{N}$.

In general, for an affine chart $\phi\colon \mathbb{A}^n \xrightarrow{\sim} G \subseteq \mathbb{K}\mathrm{P}^n$ and projectivity $f\colon \mathbb{K}\mathrm{P}^n \xrightarrow{\sim} \mathbb{K}\mathrm{P}^n$ with $f(G) \subseteq G$, the induced action on the affine space is given by

$$\phi^{-1} \circ f \circ \phi\colon \mathbb{A}^n \xrightarrow{\phi} G \xrightarrow{f} G \xrightarrow{\phi^{-1}} \mathbb{A}^n.$$

The condition $f(G) \subseteq G$ is needed to ensure compatibility.

Conversely, if given an affine transformation of the affine space $\mathbb{A}^n$, we want to find the induced action on the image of an affine chart. This ensures that the affine subspace is mapped onto itself.

Example 2.4.21. As in the example above, we can represent this action by a matrix

$$A = \begin{pmatrix} B & t \\ h^T & c \end{pmatrix}$$

with $B \in \mathrm{Mat}(n \times n, \mathbb{K}), t, h \in \mathbb{K}^n$ and $c \in \mathbb{K}$. With the trick in remark 2.4.8, we represent an affine transformation $f\colon \mathbb{A}^n \to \mathbb{A}^n, x \mapsto Cx + b$ with $C \in \mathrm{GL}(n, \mathbb{K})$ and $b \in \mathbb{K}^n$ by an $((n+1) \times (n+1))$-matrix $\tilde{C} \in \mathrm{GL}(n+1, \mathbb{K})$ with

$$\tilde{C} = \begin{pmatrix} C & b \\ 0 & 1 \end{pmatrix}$$

acting on elements of the form $(x^T, 1)^T \in \mathbb{K}^{n+1}$ with $x \in \mathbb{A}^n$. This is the same as the action of A with $B = C, t = b, h = 0$ and $c = 1$ on $G_n \subseteq \mathbb{K}\mathrm{P}^n$. We immediately see that this type of projectivity does not tilt the hyperplane at infinity since its points are mapped to points at infinity.

Generalizing this construction, we arrive at the following proposition.

Proposition 2.4.22. Let $\mathbb{K}$ be a field and $n \in \mathbb{N}$.
For a general affine chart $\phi\colon \mathbb{A}^n \xrightarrow{\sim} G \subseteq \mathbb{K}\mathrm{P}^n$ and affine transformation $f\colon \mathbb{A}^n \xrightarrow{\sim} \mathbb{A}^n$, we find that the action of f on $G = \phi(\mathbb{A}^n)$ is given by

$$\phi \circ f \circ \phi^{-1}\colon G \xrightarrow{\phi^{-1}} \mathbb{A}^n \xrightarrow{f} \mathbb{A}^n \xrightarrow{\phi} G.$$

This may be represented by some matrix if given coordinates.

Example 2.4.23. Since translations by vectors $v \in \mathbb{K}^n$ are affine transformations of the affine space $\mathbb{A}^n$, their action on the affine chart can be made more explicit. If we choose as hyperplane at infinity the hyperplane H with normal vector $h = (0, 0, \dots, 0, 1)^T$, we see that the affine part of the projective space is given by elements of the form $[x_0 : \dots : x_{n-1} : 1]$. The translation by $v \in \mathbb{K}^n$ is then given up to non-zero scalar multiples by

$$T_v = \begin{pmatrix} I_n & v \\ 0 & 1 \end{pmatrix}$$

with $I_n \in \mathrm{Mat}(n \times n, \mathbb{K})$ the identity matrix in n dimensions.

With this, we see that working with the affine subspace of a projective space is quite intuitive within this representation.

2.5 Quadratic Forms and Quadrics

In physics and mathematics, we are often interested in distances between two points, which are squared given by some polynomial of degree two in the coordinates, or sets of points which all have the same distance. A basic example would be the Euclidean metric d on $\mathbb{R}^2$ which is given by $d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3^2)}$ which is interpreted as the distance between the points x and y in $(\mathbb{R}^3, d)$. It can be represented by a diagonal matrix $E = \operatorname{diag}(1,1,1)$ in the sense that $d(x,y) = (x-y)^T E(x-y)$. It is also positive definite, meaning that it is always positive and zero only if the two points coincide. A sphere S^2 centred around $0 \in \mathbb{R}^3$ with radius $r = 1$ is then the set of points which all have the distance $r = 1$ to its centre, i. e. $S^2 = \{x \in \mathbb{R}^3 \mid d(x,0) = \sqrt{x_1^2 + x_2^2 + x_3^2} = 1\}$.

Another example from special relativity is *Minkowski spacetime* which is $\mathbb{R}^{d+1}$ with the Riemannian metric represented by $\eta = \operatorname{diag}(-1,1,\dots,1) \in \operatorname{Mat}((d+1)\times(d+1), \mathbb{R})$. Here, the form is indefinite, meaning that it can produce positive and negative distances and even zero even though the two points do not coincide. The set of points of zero distance from the origin is known as the *light cone* at the origin.

Both of these examples can be generalized in a projective setting by the use of *quadratic forms* or *homogeneous polynomials* of degree 2, which will be a replacement for the metric, to so-called (projective) *quadrics* which will be the zero locus of some quadratic form and can be considered as generalizations of conic sections. We will see that these quadrics can be easily classified over finite fields since they all can be brought into some diagonal form. Over finite fields with -1 a non-square in a projective setting, this can be adapted to find that in even projective dimensions there is actually only one type of quadrics, namely the ones of Minkowskian type.

We want to use these quadrics to define a notion of distance in our finite spacetime which is usually done via the introduction of a metric tensor $g_{\mu\nu}$. Even though affine quadrics exists, the projective setting is much more suitable for their construction. To define quadrics in a general projective space, we need to introduce quadratic forms and homogeneous polynomials which will be used to represent the quadratic forms.

Definition 2.5.1. Let $\mathbb{K}$ be a field with $\operatorname{char}(\mathbb{K}) \neq 2$ and V an n-dimensional vector space over $\mathbb{K}$ with $n \in \mathbb{N}$.

1. A **quadratic form** f is a map $f \colon V \to \mathbb{K}$ such that $f(\lambda v) = \lambda^2 f(v)$ for any $v \in V$ and $\lambda \in \mathbb{K}$.

2. A polynomial $p \in \mathbb{K}[X_0, \dots, X_{n-1}]$ in n variables is called **homogeneous of degree** $k \in \mathbb{N}$ if every non-zero term is of degree k, i. e. the sum of the exponents of each variable in this term is k.

Remark 2.5.2.

1. A homogeneous polynomial $p \in \mathbb{K}[X_0, \dots, X_{n-1}]$ of degree $k = 2$ may be represented by an $(n\times n)$-matrix $M \in \operatorname{Mat}(n\times n, \mathbb{K})$ by identifying the evaluation of p at a point $x \in \mathbb{K}^n$ with $x^T M x$, i. e. $p(x) = x^T M x$. M is then called **representation matrix of** p. This matrix may be chosen to be symmetric.

2. We can immediately see that quadratic forms on $V = \mathbb{K}^n$ may be realized by homogeneous polynomials of degree 2 in n variables. These may be represented by some matrix such that the quadratic form can be represented by said matrix.

3. A homogeneous polynomial $p \in \mathbb{K}[X_0, \ldots, X_{n-1}]$ with representation matrix $M \in \mathrm{Mat}(n \times n, \mathbb{K})$ is also represented by the symmetrized matrix $\mathrm{Sym}(M) := \frac{1}{2}(M + M^T)$ which is always symmetric. This is the case because for $x \in \mathbb{K}^n$, $p(x) = x^T M x = (x^T M^T x)^T = x^T M^T x$ since this is only a scalar. Thus, we may assume the representation matrix always to be symmetric.

Example 2.5.3. Consider the vector space $\mathbb{K}^2$ for $\mathbb{K} = \mathbb{R}$ and the quadratic form $f(x) = x_1^2 + 2x_1x_2 + x_2^2$ for $x \in \mathbb{R}^2$. To find the representation matrix $M \in \mathrm{Mat}(2 \times 2, \mathbb{R})$ of f, we look at the condition $f(x) \overset{!}{=} x^T M x$. This can be achieved by, e. g. the matrices

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ or } M = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Often it is more useful to work with a symmetric representation matrix, so we will often just chose this version for simplicity.

If we now specialize to the case of a finite field $\mathbb{F}_q$ with $q \in \mathbb{N}$ odd, we find that every quadratic form can be brought into a rather easy form, namely a diagonal form.

Theorem 2.5.4. Let $\mathbb{F}_q$ be a finite field of odd order and $f\colon \mathbb{K}^{n+1} \to \mathbb{F}_q$ be a quadratic form in $n+1$ variables, $n \in \mathbb{N}$.

Then, f is equivalent to a diagonal quadratic form of the form $a_0x_0^2 + \ldots + a_nx_n^2$ with $a_0, \ldots, a_n \in \mathbb{K}$.

Remark 2.5.5. Since quadratic forms f in n variables over some finite field $\mathbb{F}_q$ of odd order may be represented by an $(n \times n)$-matrix $M \in \mathrm{Mat}(n \times n, \mathbb{F}_q)$, the theorem above tells us that this matrix may be brought into a diagonal form $\mathrm{diag}(a_0, \ldots, a_n)$ by a transformation of the form $M \mapsto (S^{-1})^T M S^{-1}$ for some $S \in \mathrm{GL}(n, \mathbb{F}_q)$ since the value of $f(x) = x^T M x$ should not change under the transformation $x \mapsto Sx$.

In the following, we may w.l.o.g. assume that the quadratic forms over $\mathbb{F}_q, q$ odd, are already diagonal.

Some of the terminology used in the study of matrices can be adapted to quadratic forms by the use of their representation matrix.

Definition 2.5.6. Let $\mathbb{K}$ be a field with $\mathrm{char}(\mathbb{K}) \neq 2$, $f\colon \mathbb{K}^n \to \mathbb{K}$ be a quadratic form in $n \in \mathbb{N}$ variables and $M \in \mathrm{Mat}(n \times n, \mathbb{K})$ its symmetric representation matrix.

1. f is called **degenerate** if M is degenerate, i. e. $\det(M) = 0$ or, if M is in diagonal form, at least one of the diagonal elements is zero.

2. The **determinant** $\det(f)$ of the quadratic form f is the determinant of M, i. e. $\det(f) := \det(M)$.

In the following, we will often assume the quadratic forms to be non-degenerate for simplicity.

Now, consider a non-degenerate quadratic form $f\colon \mathbb{F}_q^{n+1} \to \mathbb{F}_q$ with $q \in \mathbb{N}$ an odd prime power and an equation of the form $f(x) = f(x_0, \ldots, x_n) = b$ with $b \in \mathbb{F}_q$. Since

$\mathbb{F}_q$ is a finite field and is, thus, by definition finite, we can count the number of solutions of such an equation. These results will later be useful to find the number of points in a quadric as the zero locus of a non-degenerate quadratic form in some projective space.

Since there exist non-squares in a finite field $\mathbb{F}_q$, the number of solutions will depend on the nature of the pre-factors and of b. To account for that, we have already defined the Legendre or Jacobi symbol in 2.2.32. The following theorem [9, p. 282f] provides a solution to the question of the number of solutions of an equation of the form $f(x_0, \dots, x_n) = b$ for the cases of n being even or odd. It will make use of the Jacobi symbol in the form of the *quadratic character* $\eta\colon \mathbb{F}_q \to \{-1, 0, 1\}$ of $\mathbb{F}_q$ which coincides with the Jacobi symbol.

Theorem 2.5.7. Let $\mathbb{F}_q$ be the finite field of odd order $q \in \mathbb{N}$, $f\colon \mathbb{F}_q^n \to \mathbb{F}_q$ be a non-degenerate quadratic form in n variables and $b \in \mathbb{F}_q$.

1. The number of the solutions of the equation $f(x_1, \dots, x_n) = b$ with n **even** is given by
$$q^{n-1} + \nu(b) q^{(n-2)/2} \eta\left((-1)^{n/2}\Delta\right),$$
where $\nu\colon \mathbb{F}_q \to \mathbb{Z}$, $\nu(b) = -1$ for $b \in \mathbb{F}_q \setminus \{0\}$ and $\nu(b) = q - 1$ for $b = 0$, η is the quadratic character of $\mathbb{F}_q$, which coincides with the Jacobi symbol, and $\Delta = \det(f)$.

2. The number of the solutions of the equation $f(x_1, \dots, x_n) = b$ with n **odd** is given by
$$q^{n-1} + q^{(n-1)/2} \eta\left((-1)^{(n-1)/2} b\Delta\right),$$
with the same definitions as above.

We are now able to define (projective) quadrics as the zero locus of some quadratic form on a projective space. The projective setting is especially useful since both introductory examples of the sphere and the light cone can be treated in the same manner.

Definition 2.5.8. Let $\mathbb{K}$ be a field with $\mathrm{char}(\mathbb{K}) \neq 2$, $n \in \mathbb{N}$ a natural number and $f\colon \mathbb{K}^{n+1} \to \mathbb{K}$ a quadratic form with representation matrix $M \in \mathrm{Mat}((n+1)\times(n+1), \mathbb{K})$.

The **(projective) quadric** $Q \subseteq \mathbb{K}\mathrm{P}^n$ associated with the quadratic form f is the zero locus of f in n-dimensional projective space $\mathbb{K}\mathrm{P}^n$, i. e. $Q = \{[x] \in \mathbb{K}\mathrm{P}^n \mid f(x) = 0\} \subseteq \mathbb{K}\mathrm{P}^n$. M is then also called the **representation matrix** of the quadric Q, meaning that $Q = \{[x] \in \mathbb{K}\mathrm{P}^n \mid x^T M x = 0\}$. If f is degenerate, the quadric Q will also be called **degenerate**.

Remark 2.5.9.

1. Note that in this definition the quadratic form f, represented by some homogeneous polynomial, is necessary and cannot be replaced by an ordinary polynomial of degree two since its homogeneity is needed for it to be well-defined on projective space $\mathbb{K}\mathrm{P}^n = \mathbb{K}^{n+1}/\sim$. For $y \in [x] \in \mathbb{K}\mathrm{P}^n$ with $f(x) = 0$, we find that $y = \lambda x$ for some $\lambda \in \mathbb{K}^\times$. With the properties of a quadratic form, we immediately find

that $f(y) = f(\lambda x) = \lambda^2 f(x) = 0$ such that this equation is independent of the representative of the equivalence class $[x]$. This would not be the case if f was not homogeneous.

2. (Projective) quadrics are a special case of **projective varieties** which are usually defined over some algebraically closed field, meaning that every polynomial in one variable with coefficients in this field has at least one root, as the zero locus (as a subset of n-dimensional projective space) of a finite family of homogeneous polynomials in $n+1$ variables with some restrictions on the choice of these polynomials for non-degeneracy, i. e. $\{[x] \in \mathbb{K}\mathrm{P}^n \mid f_1(x) = \ldots = f_m(x) = 0\}$ with $f_1, \ldots, f_m \in \mathbb{K}[X_0, \ldots, X_n]$ homogeneous polynomials in $n+1$ variables and $m \in \mathbb{N}$. There is a rich theory about varieties, known as algebraic geometry.

3. In the case of $\mathbb{K} = \mathbb{F}_p$ with $p \in \mathbb{N}$ prime and $p \equiv 3 \pmod 4$, it can be shown that in even dimensions n every non-degenerate quadratic form can be brought into one of two (equivalent) canonical forms $M = \mathrm{diag}\,(-1, 1, \ldots, 1, \pm g^2)$ with $g \in \mathbb{F}_p^\times$ [11]. Following the notion of special relativity, we will call these canonical forms M the **(standard) Minkowski forms** if $g = 1$, denoted by $\eta^\pm = \mathrm{diag}(-1, 1, \ldots, 1, \pm 1)$. Quadrics associated with η^+ will correspond to time-like and quadrics associated with η^- to space-like neighbours of unit distance in the affine space. Points at infinity of these quadrics will correspond to light-like neighbours.

Example 2.5.10. Let us consider the examples from the beginning of this section in this new context:

1. The sphere $S^2 = \{x \in \mathbb{R}^3 \mid x^T x = 1\}$ can be made a projective quadric $Q \subseteq \mathbb{R}\mathrm{P}^3$ by the introduction of affine coordinates on $\mathbb{R}\mathrm{P}^3$, meaning that S^2 is actually the affine part of this projective quadric Q with respect to some hyperplane. We choose these coordinates to be $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3 \mapsto [x_1 : x_2 : x_3 : 1] \in \mathbb{R}\mathrm{P}^3$. Then, the projective quadric is given by $Q = \{[x] \in \mathbb{R}\mathrm{P}^3 \mid x^T M x = 0\}$ with $M = \mathrm{diag}(1, 1, 1, -1)$. Indeed, for an affine element $y = [x_1 : x_2 : x_3 : 1] \in \mathbb{R}\mathrm{P}^3$ we find that the condition of y being an element of Q is given by $y^T M y = x_1^2 + x_2^2 + x_3^2 - 1 = 0$ which is equivalent to $x^T x = 1$ for $x = (x_1, x_2, x_3)^T$. Thus, we have found an embedding $S^2 \hookrightarrow Q$ as the affine part of Q.

2. The light cone $L = \{x \in \mathbb{R}^{d+1} \mid x^T \eta x = -x_0^2 + x_1^2 + \ldots + x_d^2 = 0\} \subseteq \mathbb{R}^{d+1}$ can be seen as the points at infinity of the projective quadric $\{[x] \in \mathbb{R}\mathrm{P}^{d+1} \mid x^T \eta^\pm x = 0\}$ with the standard Minkowski forms $\eta^\pm$. Indeed, we find that for points $p \in \mathbb{R}\mathrm{P}^{d+1}$ at infinity, which are of the form $p = [y_0 : y_1 : \ldots : y_d : 0]$, that $p^T \eta^\pm p = -y_0^2 + y_1^2 + \ldots + y_d^2 \pm 0^2 = -y_0^2 + y_1^2 + \ldots + y_d^2 = 0$ if $(y_0, y_1, \ldots y_d) \in L$, independently of the sign of $\eta^\pm$. This shows, that the light cone can be embedded into these projective quadrics.

The two examples show that objects which represent points of a given distance can be generalized to projective quadrics where they can be treated in the same fashion in a projective setting. Thus, projective quadrics offer us a well-behaved meaning of points of the same distance to some central point.

Later, we will use these quadrics as part of a **(bi-)quadric field** which associates to every point in our projective spacetime two quadrics with certain compatibility restrictions [11, 8]. These two quadrics should offer the feature that every line through the *centre* of the bi-quadric in the projective space has exactly two intersection with this bi-quadric, but there is no general, mathematical sound proof that this is always possible as of yet.

For the most time, we will focus on one part of this bi-quadric and assume it to be non-degenerate. Often we will assume it to be in a diagonal form, or even in the standard Minkowski forms, at one point and then transformed by some projectivity at another point.

There are different ways to define the notion of the centre of a (bi-)quadric. One is rather geometrical with the centre being a point in the projective space and not on the quadric such that every line through this centre has exactly two intersections with the bi-quadric. But, this is rather cumbersome to compute. Thus, we will give a more algebraic definition with the use of a hyperplane at infinity.

Definition 2.5.11. Let $Q \in \mathbb{K}\mathrm{P}^n$ be a non-degenerate projective quadric over the field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) \neq 2$, $n \in \mathbb{N}$ with symmetric representation Matrix $M \in \mathrm{Mat}((n+1) \times (n+1), \mathbb{K})$, and let $h_\infty \in \mathbb{K}^{n+1}$ be the normal vector of a chosen hyperplane at infinity H_∞.

The **centre** $c \in \mathbb{K}\mathrm{P}^n$ of the quadric Q with respect to H_∞ is given by $c = M^{-1} h_\infty$ up to non-zero scalar multiples.

Example 2.5.12. For the usual choice of $h_\infty = (0, \dots, 0, 1)^T \in \mathbb{K}^{n+1}$ and the standard Minkowski forms $\eta^\pm$, we find that the centre $c \in \mathbb{K}\mathrm{P}^n$ is given by $c = (\eta^\pm)^{-1} h_\infty = \eta^\pm h_\infty = [0 : \dots : 0 : \pm 1] = [0 : \dots : 0 : 1]$.

Since all quadratic forms over the finite field $\mathbb{F}_p$ with $p \equiv 3 \pmod 4$ in even dimension can be brought into the standard Minkowski forms by a transformation of the representation matrix A of the quadratic form of the form $A \mapsto A' = (S^{-1})^T A S^{-1}$, the (bi-)quadric field can be described by a field of the transformation matrices $S \in \mathrm{PGL}(n+1, \mathbb{F}_p)$ such that the centre and location of the quadric corresponding to A is $c_A = [0 : 0 : \dots : 0 : 1]$ with our usual choice of hyperplane at infinity. The centre of the quadric corresponding to A' need not be the point where this quadric is located in terms of the (bi-)quadric field.

A special case of this transformation behaviour is given by the (bi-)quadric field which is constructed by only using translations from one point to the other in the affine subspace as transformation matrices such that for $c_A = [0 : \dots : 0 : 1]$ the (bi-)quadric corresponds to the standard Minkowski forms $\eta^\pm$. Here, the centre and the location of the quadric coincide. This will be called a *flat (force-free) spacetime* which will be defined in the next chapter.

Since the finite field used in the constructions above is by definition finite, we can count the number of elements of a non-degenerate quadric if it is represented by a diagonal form by the use of 2.5.7 and the decomposition $\mathbb{K}\mathrm{P}^n = \{[x_0 : \ldots : x_{n-1} : 1]\} \cup \{[y_0 : \ldots : y_{n-2} : 1 : 0]\} \cup \ldots \{[a : 1 : 0 : \ldots]\} \cup \{[1 : 0 : \ldots]\}$. We will refer to [15] for the proof in four dimensions which can be easily generalized to even dimensions.

Proposition 2.5.13. Let $\mathbb{F}_q$ be a finite field of odd order $q \in \mathbb{N}$ and $Q \subseteq \mathbb{F}_q\mathrm{P}^n, n \in \mathbb{N}$ even, a non-degenerate projective quadric represented by a diagonal matrix of the form $\mathrm{diag}(a_0, \ldots, a_n)$.

Then, the number of elements of Q is independent of $a_0, \ldots a_n$ and is given by

$$|Q| = q^{n-1} + q^{n-2} + \ldots + q + 1.$$

In particular, for $n = 4$, we have $|Q| = q^3 + q^2 + q + 1$.

Remark 2.5.14.

1. In odd dimensions, there is also a dependence on the determinant of the quadratic form corresponding to the quadric. In three dimensions, the number of elements is given by $q^2 + \eta(a_0a_1a_2a_3)q + q + 1$ with η the quadratic character of $\mathbb{F}_q$. This is in complete analogy in higher dimensions.

2. Note that in even dimensions the number of elements of a non-degenerate quadric in n-dimensional projective space is equivalent to the number of elements in $(n-1)$-dimensional projective space.

The number of points of a quadric in a certain hyperplane will be useful later in the study of the intersection of two quadrics as an idea for the implementation of gauge transformations. This number may be calculated by adapting the methods from [15] to this situation.

3 Finite Projective Physics

In most of physics, a matter-based ontology is used where the main building blocks are matter particles and their trajectories on a background spacetime. The trajectories of the particles are given by solutions of the equations of motion derived from some suitable kinematic Quantum Field Theory on a curved spacetime. This spacetime is not only a background for the motion of the particles but is also affected by them via the Einstein equations of General Relativity. Here lies one of the problems of the unification of the theory of General Relativity and Quantum Field Theory: Matter is described by quantum theoretical equations on a background spacetime, spacetime in General Relativity as a classical object, but both should be able to properly interact. Thus, many theories like String Theory or Loop Quantum Gravity have been developed to resolve this issue and many others, but, up to now, many different problems remain and no widely-accepted unification of Quantum Field Theory and General Relativity exists.

In stark contrast to this matter-based approach, we want to present and review in the following some of the foundations of Finite Projective Physics which can be seen as an attempt to unify gravity and quantum theory by using a projective space over some finite field as an embedding space for sequences of events in an event-based ontology where the essential building block are *events of creation* and their *sequences of succession*.

3.1 Foundations of Finite Projective Physics

In this new attempt of Finite Projective Physics [13, 12], we try to resolve some of the issues described above by using an event-based ontology instead of the matter-based approach. Here, the main objects are 'events of creation' which do not lead to any existing objects per se, only sequences of events which are related to each other by some succession relation do. The idea should be like this: If an object exists for some time at some point in space, there is for every point in spacetime which is covered by the existence of the object an event of creation of this object in a causal relation. These events form a sequence of succession since every event is caused by its *predecessor* and causes its *successor* in this sequence. This causal succession relation should be thought of as a discrete notion of 'internal time' or 'proper time' of this sequence, giving a notion of 'before' and 'after'. The spacetime itself is given by all possible events.

This approach is suitable for the description of special relativistic physics since it differentiates coordinate time from proper time. It is also suitable for quantum theoretical models since Heisenberg's Uncertainty Principle implies that we cannot determine arbitrarily small intervals of spacetime. Thus, a discretization of spacetime at the level of Planck length and time seems to be reasonable and because of the finiteness of the (observable) universe only a finite amount of events suffices.

These sequences of events can lead to new sequences or they can interact with other sequences, giving rise to other sequences. These networks of sequences can give a definition of dimension and give rise to a geometry as an embedding space. This can be thought of as a background on which the physical spacetime is described. To remember the form of the embedding, we apply *occupation numbers* to the points of the ambient space such that each point of a sequence is labelled by a 1 and every other point not part of any sequence has label 0. Hence, we can work with the embedding space and only later consider the sequences of succession.

As it turns out, we can use a projective space $\mathbb{F}_q\mathrm{P}^n$ for some prime power $q \in \mathbb{N}$ and $n \in \mathbb{N}$ as an embedding space and will, in the following, only work with this space as a model of spacetime. The notion of predecessor and successor is translated to an intersection of a line with a (bi)-quadric which also encapsulates the *law of inertia* as in Newton's first law. We will make this more precise in our description of Newtonian mechanics in Finite Projective Physics later.

3.1.1 (Bi-)Quadric Fields and Finite Projective Spacetimes

We start with the definition of a (bi-)quadric field as an assignment of a (bi-)quadric to every point in a projective space.

Definition 3.1.1. Let $\mathbb{F}_q$ be a finite field of odd order $q \in \mathbb{N}$ and $n \in \mathbb{N}$. Furthermore, let $(\mathrm{B})\mathrm{Q}^n_{nd}$ be the set of all possible non-degenerate projective (bi-)quadrics on $\mathbb{F}_q\mathrm{P}^n$ such that one part of the bi-quadric is equivalent to η^+ and the other one is equivalent to η^-.

1. A (non-degenerate) **(bi-)quadric field** $Q\colon \mathbb{F}_q\mathrm{P}^n \to (\mathrm{B})\mathrm{Q}^n_{nd}$ is an assignment $p \mapsto Q(p)$ of a non-degenerate (bi-)quadric $Q(p) \in (\mathrm{B})\mathrm{Q}^n_{nd}$ to every point $p \in \mathbb{F}_q\mathrm{P}^n$.

2. A (bi-)quadric field $Q\colon \mathbb{F}_q\mathrm{P}^n \to (\mathrm{B})\mathrm{Q}^n_{nd}$ is called **pre-causal** if, additionally, for all $b \in Q(p)$ we have $p \in Q(b)$, i. e. p is also an element of all (bi-)quadrics located at the points of the (bi-)quadric located at p, and one part of the (bi-)quadric at each point can be brought into the standard Minkowski form corresponding to η^+.

3. A pre-causal (bi-)quadric field $Q\colon \mathbb{F}_q\mathrm{P}^n \to (\mathrm{B})\mathrm{Q}^n_{nd}$ is called **causal** if in addition $p \in Q(b)^+$ for all $b \in Q(p)^+$ where $q(x)^+$ denotes the part of the (bi-)quadric equivalent to η^+, i. e. time-like neighbours.

Remark 3.1.2.

1. In principle, it would be possible to also include degenerate quadrics. We will exclude those cases for simplicity.

2. Note that since a projective quadric on $\mathbb{F}_p\mathrm{P}^n$ for a prime field $\mathbb{F}_p$ with n even and $p \equiv 3 \pmod 4$ can be brought into one of the standard Minkowski forms represented by $\eta^\pm$, we can choose coordinates such that a quadric field is described

by the transformation matrices $S \in \mathrm{PGL}(n+1, \mathbb{F}_p)$ that transform the standard Minkowski form into our desired representation matrix of the quadric at the given point. This leads, in the case of a bi-quadric field, to an assignment of the form $\mathbb{F}_p\mathrm{P}^n \to \mathrm{PGL}(n+1, \mathbb{F}_p) \times \mathrm{PGL}(n+1, \mathbb{F}_p)$ for each part of the bi-quadric with certain compatibility conditions.

3. A (pre-)causal (bi-)quadric field should be thought of in the sense of predecessors and successors of an event which are given by points on a quadric located at the event. The additional condition now tells us that the event itself is, indeed, a predecessor of its successors and a successor of its predecessors if given some causal direction determining the order of such events. The notion of a causal (bi-)quadric field guaranties that time-like neighbours stay time-like even when switching from one to the other.

A special case of a (bi-)quadric field is a *flat* and *force-free* (bi-)quadric field. Here, we want to consider the nth standard affine chart of $\mathbb{F}_q\mathrm{P}^n$, i. e. an embedding of the affine space $\mathbb{A}^n \hookrightarrow G_n \subseteq \mathbb{F}_q\mathrm{P}^n$ with respect to the hyperplane at infinity with normal vector $(0, \ldots, 0, 1)^T \in \mathbb{F}_q^{n+1}$. The origin O of $\mathbb{A}^n$ in this chart is given in homogeneous coordinates by $c = [0 : \ldots : 0 : 1] \in \mathbb{F}_q\mathrm{P}^n$. Every point $x \in \mathbb{A}^n$ can be reached via a translation $T_{O\to x}\colon \mathbb{A}^n \to \mathbb{A}^n$ from the origin $O \in \mathbb{A}^n$ to x. The corresponding projectivity acting on the affine chart will be denoted $T_{c\to x}$.

The flat and force-free (bi-)quadric field is then given at c as the (bi-)quadric corresponding to $\eta^{\pm}$, respectively. At any other point $y \in G_n \subseteq \mathbb{F}_q\mathrm{P}^n$ of the affine subspace G_n, the (bi-)quadric field is given by the translation of the (bi-)quadric at c via $T_{c\to y}$, i. e. the representation matrices of the (bi-)quadric at y are given by $(S(y)^{-1})^T\eta^{\pm}S(y)^{-1}$ where $S(y) \in \mathrm{PGL}(n+1, \mathbb{F}_q)$ is the representation matrix of $T_{c\to y}$. Herein, the centre of the (bi-)quadrics and their location coincide. This property leads to the term 'force-free' which will become clearer after the introduction of forces in Newtonian mechanics.

To summarize these notions, we give the following definition.

Definition 3.1.3. Let $\mathbb{F}_q$ be a finite field of odd order $q \in \mathbb{N}$ and $n \in \mathbb{N}$. Furthermore, let $(\mathrm{B})\mathrm{Q}^n_{nd}$ be the set of all possible non-degenerate projective (bi-)quadrics on $\mathbb{F}_q\mathrm{P}^n$ and let $G \subseteq \mathbb{F}_q\mathrm{P}^n$ be the image of an affine chart $\mathbb{A}^n \to G$ with respect to the hyperplane $H_\infty \subseteq \mathbb{F}_q\mathrm{P}^n$ with normal vector $h_\infty \in \mathbb{F}_q^{n+1}$.

1. A (bi-)quadric field $Q\colon \mathbb{F}_q\mathrm{P}^n \to (\mathrm{B})\mathrm{Q}^n_{nd}$ is called **flat** with respect to H_∞ if its restriction onto the affine subspace G is given by translations of the (bi-)quadric corresponding to the standard Minkowski form $\eta^{\pm}$ with respect to H_∞, i. e. the representation matrices of the (bi-)quadric $Q(p)$ for some $p \in G$ is given by $(S^{-1})^T\eta^{\pm}S^{-1}$ where $S \in \mathrm{PGL}(n+1, \mathbb{F}_q)$ is the representation matrix of the projectivity corresponding to some translation with respect to H_∞.

2. A (bi-)quadric field $Q\colon \mathbb{F}_q\mathrm{P}^n \to (\mathrm{B})\mathrm{Q}^n_{nd}$ is called **force-free** if the centre of the (bi-)quadric $Q(p)$ for any $p \in \mathbb{F}_q\mathrm{P}^n$ is equal to p.

The force-free and flat case can be made more explicit.

Example 3.1.4 (Force-free and flat (bi-)quadric field)**.** We consider for a finite field $\mathbb{F}_q$ with $q \in \mathbb{N}$ odd the projective space $\mathbb{F}_q\mathrm{P}^n$ with $n \in \mathbb{N}$ and with the nth standard affine chart $\mathbb{A}^n \to G_n \subseteq \mathbb{F}_q\mathrm{P}^n$ as definition of our hyperplane at infinity H_∞, i. e. $h_\infty = (0, \ldots, 0, 1)^T \in \mathbb{F}_q^{n+1}$. We want to find a force-free and flat (bi-)quadric field $Q\colon \mathbb{F}_q\mathrm{P}^n \to (\mathrm{B})\mathrm{Q}^n_{nd}$.

Let us consider the point $c = [0 : \ldots : 0 : 1] \in \mathbb{F}_q\mathrm{P}^n$. As in example 2.5.12, we find that, for the standard Minkowski forms $\eta^\pm$, c is the centre with respect to H_∞. Thus, we define $Q(c)$ to be the (bi-)quadric corresponding to the standard Minkowski forms $\eta^\pm$. For a point $x = [x_0 : x_1 : \ldots : x_{n-1} : 1] =: [\tilde{x} : 1] \in G_n$, the translation $T_{c\to x}$ from c to x is represented by the matrix

$$T(x) \coloneqq T_{c\to x} = \begin{pmatrix} I_n & \tilde{x} \\ 0 & 1 \end{pmatrix} \in \mathrm{PGL}(n+1, \mathbb{F}_q)$$

with $I_n \in \mathrm{GL}(n, \mathbb{F}_q)$ the identity matrix in n dimensions, i. e. $T_{c\to x}c = x$. At x, we define $Q(x)$ to be the (bi-)quadric corresponding to $M^\pm(x) = (T(x)^{-1})^T \eta^\pm T(x)^{-1}$. Explicitly, this is given by the matrices

$$\begin{aligned} M^\pm(x) &= \begin{pmatrix} I_n & 0 \\ -\tilde{x}^T & 1 \end{pmatrix} \begin{pmatrix} \eta & 0 \\ 0 & \pm 1 \end{pmatrix} \begin{pmatrix} I_n & -\tilde{x} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \eta & -\eta\tilde{x} \\ -\tilde{x}^T\eta & \tilde{x}^T\eta\tilde{x} \pm 1 \end{pmatrix} = \begin{pmatrix} \eta & -\eta\tilde{x} \\ -\tilde{x}^T\eta & x^T\eta^\pm x \end{pmatrix} \end{aligned} \tag{3.1}$$

with $\eta = \mathrm{diag}(-1, 1, \ldots, 1) \in \mathrm{Mat}(n \times n, \mathbb{F}_q)$.

The centre of $Q(x)$ is given by

$$(M^\pm(x))^{-1} h_\infty = ((T(x)^{-1})^T \eta^\pm T(x)^{-1})^{-1} h_\infty = T(x)\eta^\pm T(x)^T h_\infty = \pm x = x \in \mathbb{F}_q\mathrm{P}^n$$

since $(\eta^\pm)^2 = I_{n+1}$ and

$$T(x)\eta^\pm T(x)^T = \begin{pmatrix} I_n & \tilde{x} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \eta & 0 \\ 0 & \pm 1 \end{pmatrix} \begin{pmatrix} I_n & 0 \\ \tilde{x}^T & 1 \end{pmatrix} = \begin{pmatrix} \eta \pm \tilde{x}\tilde{x}^T & \pm\tilde{x} \\ \pm\tilde{x}^T & \pm 1 \end{pmatrix}.$$

Multiplication with h_∞ then essentially selects only the last column of this matrix.

Note that since we do not mix the types of $\eta^\pm$ in this construction and points in the quadrics are given by a translation of the points of the standard Minkowski quadric at c, we immediately conclude that this (bi-)quadric field is causal.

With the definition of a (bi-)-quadric field, we are now able to define a *projective spacetime* over a finite field $\mathbb{F}_q$.

Definition 3.1.5. Let $\mathbb{F}_q$ be a finite field of odd order $q \in \mathbb{N}$, $n \in \mathbb{N}$ and $(\mathrm{B})\mathrm{Q}^n_{nd}$ be the set of all possible non-degenerate projective (bi-)quadrics on $\mathbb{F}_q\mathrm{P}^n$ such that one part of the bi-quadric is equivalent to η^+ and the other one is equivalent to η^-.

1. A (finite) **projective spacetime** of dimension n is a tuple $(\mathbb{F}_q\mathrm{P}^n, Q)$ of an n-dimensional projective space $\mathbb{F}_q\mathrm{P}^n$ over $\mathbb{F}_q$ and a non-degenerate, causal (bi-)quadric field $Q\colon \mathbb{F}_q\mathrm{P}^n \to (\mathrm{B})\mathrm{Q}^n_{nd}$.

2. A projective spacetime $(\mathbb{F}_q\mathrm{P}^n, Q)$ is called **flat** if Q is flat.

3. A projective spacetime $(\mathbb{F}_q\mathrm{P}^n, Q)$ is called a **vacuum spacetime** or **force-free** if Q is force-free.

4. An element of this projective spacetime is called an **event**.

3.1.2 Finite Relativistic Newtonian Mechanics - Kinematics and Dynamics

Up to now, we only considered the spacetime upon which the dynamics and kinematics of events and consequently of matter should take place. It is not clear yet how kinematics and dynamics work in Finite Projective Physics. We will see that both are adapted from the laws of relativistic Newtonian mechanics.

In the following, we will work with natural units such that the speed of light $c = 1$ and adapt the notion of of a four-dimensional spacetime from Special Relativity.

From Special Relativity to Finite Kinematics and Dynamics In Special Relativity, the kinematics of particles in a particular inertial reference frame are described by their four-vector of position $x = (t, \vec{x}^T)^T \in \mathbb{R}^{1+3}$ in flat Minkowski spacetime $(\mathbb{R}^{1+3}, \eta)$, with $\eta = \mathrm{diag}(-1, 1, 1, 1)$, the Minkowski metric, as a function of proper time $\tau \in \mathbb{R}$ and by their four-velocity $u = (u^0, \vec{u}^T)^T \in \mathbb{R}^{1+3}$ at some position $x(\tau)$ subject to the condition

$$\dot{x}(\tau) = u(x(\tau)) =: u(\tau)$$

where the dot $\dot{-}$ denotes the derivative with respect to τ, i. e. the 4-velocity describes the change of position with respect to τ.

This means that the trajectory of a particle is described by two differentiable functions of proper time $x, u\colon \mathbb{R} \to \mathbb{R}^{1+3}$ with the relation $\dot{x} = u$. Note that the position function x does not map to the same space as the velocity function to be precise. This is because the velocity function actually maps to the tangent space $T_{x(\tau)}\mathbb{R}^{1+3}$ at the point $x(\tau)$ which can be identified with $\mathbb{R}^{1+3}$. Intrinsically, these two quantities are fundamentally different since velocities translate between two (infinitesimally) neighbouring points given by the definition of the derivative

$$u(\tau) = \dot{x}(\tau) = \frac{\mathrm{d}x}{\mathrm{d}\tau}(\tau) = \lim_{h\to 0} \frac{x(\tau + h) - x(\tau)}{h}.$$

Since the 0th component x^0 of the four-vector of position x is the time-like component, i. e. $x^0 = t$, the 0th component of the four-velocity describes the relation between the temporal coordinate t in spacetime and proper time τ, known as the **Lorentz factor**

$$\gamma = u^0 = \dot{t} = \frac{\mathrm{d}t}{\mathrm{d}\tau}.$$

Another interesting fact is that the Minkowski-pseudo-norm of the four-velocity is always constant and shows that these velocities are all time-like, i. e.

$$\eta(u,u) = u^T \eta u = -(u^0)^2 + (u^1)^2 + (u^2)^2 + (u^3)^2 = -\gamma^2 + ||\vec{u}||^2 = -1$$

with $||\vec{u}||^2 = (u^1)^2 + (u^2)^2 + (u^3)^2$ the norm of the spatial velocity. Using the relation $\vec{u} = \gamma\vec{v}$ with $v = \frac{\mathrm{d}x}{\mathrm{d}t}$ and re-arranging leads to the widely known expression for the Lorentz factor

$$\gamma = \frac{1}{\sqrt{1 - ||\vec{v}||^2}} = \frac{1}{\sqrt{1-\beta^2}}$$

with $\beta = ||\vec{v}||$ in natural units.

The dynamics of a particle described in an inertial reference frame are subject to the relativistic analogues of Newton's Laws of Motion. The first law states roughly that a body does not change its state of motion, meaning that it either stays in rest or stays moving with constant velocity on a straight line, unless it is subject to a force which changes its state of motion. This leads to the concept of *inertia*. This can also be seen as a definition of 'straight lines' since force-free motion with a non-zero velocity should stay on these straight lines which could then be measured.

The second law describes the relation between the change of momentum of a body and the force applied to it. For a body with mass $m \in \mathbb{R}$, the 4-momentum $p \in \mathbb{R}^{1+3}$ is given by $p = mu$ with the four-velocity u. If a four-force $F \in \mathbb{R}^{1+3}$ is applied on the mass m, the second law of motion tells us that

$$F = \dot{p} = \frac{\mathrm{d}(mu)}{\mathrm{d}\tau}.$$

If the mass m is constant with respect to τ, we find that $F = m\dot{u} = ma$ with the four-acceleration $a = \dot{u}$.

The solutions to the differential equation $\dot{x} = u$ and Newton's second law of motion with suitable initial conditions give rise to the dynamics of Special Relativity.

To adapt relativistic kinematics and dynamics and these laws of motion in a finite projective spacetime in an event-based ontology, we make the following observations.

Observation 1 The description of proper time should inherit a finite time scale of the form $\Delta\tau = 1$. This may be fulfilled by the ring of integers $\mathbb{Z}$ or a finite field $\mathbb{F}_q$ since both are generated by 1. This will be the basis of the step from one event to the next one.

Observation 2 Velocities should work like translation vectors from one event to the following. Thus, we define a *backwards velocity* $u^- \in \mathbb{F}_q^4$ from the predecessor to the current event and a *forward velocity* $u^+ \in \mathbb{F}_q^4$ from the current event to its successor. It should be noted that the backward velocity at some event should coincide with the forward velocity of its predecessor. The *mean velocity* $u_0 \in \mathbb{F}_q^4$ will be defined as

$$u_0 := \frac{u^+ + u^-}{2} \in \mathbb{F}_q^4.$$

It should satisfy the condition $u_0^T \eta u_0 = -1$, i. e. u_0 is time-like.

Observation 3 If no force is applied, three successive events should lie on a line in projective space to enforce the law of inertia, i. e. an event, its predecessor and its successor are on a line if $F = 0$. Furthermore, the successor of each event should be a time-like neighbour, i. e. be an element of the time-like quadric at the point of the event. This means that the successor of an event lies on the intersection of the time-like quadric with the line from the predecessor to the event itself.

Observation 4 If a force F is applied, three successive events should no longer lie on one line which means that the event in the middle seems to be shifted off of the straight line between its predecessor and its successor. We can still require this line to have the centre of the quadric at the event itself on it. This means that the force F leads to a translation of the centre of the quadric away from the location of the quadric. Thus, the translation vector should depend on the force F and, in accordance with Newton's second law of motion, on the mass of the particle described by this chain of events. A successor of an event should still be a time-like neighbour of this event and should lie on the intersection between the time-like quadric and the line between the predecessor event to the centre of the quadric located at the event itself.

Observation 5 In a four-dimensional spacetime, a triple of events always lies in some affine chart which may not be unique. Thus, translation with respect to some hyperplane at infinity is defined locally.

We summarize these observations in the following definition of finite relativistic Newtonian mechanics for arbitrary dimensions $n \in \mathbb{N}$.

Definition 3.1.6. Let $\mathbb{F}_q$ be a finite field of odd order $q \in \mathbb{N}$ and let $(\mathbb{F}_q\mathrm{P}^n, Q)$ be a finite projective spacetime of dimension $n \in \mathbb{N}$. Let $c(Q(y)) \in \mathbb{F}_q\mathrm{P}^n$ denote the centre of the (bi-)quadric $Q(y)$ at $y \in \mathbb{F}_q\mathrm{P}^n$.

1. The **trajectory** of a body with mass $m \in \mathbb{F}_q$ is a map $x\colon \mathbb{Z} \to \mathbb{F}_q\mathrm{P}^n$ such that for all $\tau \in \mathbb{Z}$ we have $x(\tau+1), x(\tau-1) \in Q(x(\tau))^+$ and $x(\tau+1) \in (x(\tau-1) \vee c(Q(\tau)))$, i. e. the predecessor and the successor of $x(\tau)$ should be time-like neighbours of $x(\tau)$ and the successor should lie on the line connecting the predecessor and the centre of the quadric at $x(\tau)$, thus, enforcing the law of inertia.

2. If an affine chart $\mathbb{A}^n \to G(x(\tau)) \subseteq \mathbb{F}_q\mathrm{P}^n$ with respect to the hyperplane $H_\infty(x(\tau))$ is chosen at the point $x(\tau)$ of the trajectory $x\colon \mathbb{Z} \to \mathbb{F}_q\mathrm{P}^n$ such that $x(\tau-1), x(\tau)$ and $x(\tau+1)$ lie in the affine subspace $G(x(\tau)) \subseteq \mathbb{F}_q\mathrm{P}^n$, we define the **backward velocity** $u^-(x(\tau)) \in \mathbb{F}_q^n$ as the translation vector with respect to $H_\infty(x(\tau))$ from $x(\tau-1)$ to $x(\tau)$ and the **forward velocity** $u^+(x(\tau)) \in \mathbb{F}_q^n$ as the translation vector with respect to $H_\infty(x(\tau))$ from $x(\tau)$ to $x(\tau+1)$.

3. The **mean velocity** $u_0(x(\tau)) \in \mathbb{F}_q^n$ at a point $x(\tau)$ of the trajectory $x\colon \mathbb{Z} \to \mathbb{F}_q\mathrm{P}^n$ is given by
$$u_0(x(\tau)) = \frac{u^+(x(\tau)) + u^-(x(\tau))}{2}. \tag{3.2}$$

It is subject to the condition $u_0^T \eta u_0 = -1$ with $\eta = \mathrm{diag}(-1, 1, \dots, 1) \in \mathrm{Mat}(n \times n, \mathbb{F}_q)$.

4. If a force field $F\colon \mathbb{F}_q\mathrm{P}^n \to \mathbb{F}_q^n$ is applied to a body with mass $m \in \mathbb{F}_q^\times$ with respect to some hyperplane $H_\infty \subseteq \mathbb{F}_q\mathrm{P}^n$, the trajectory $x\colon \mathbb{Z} \to \mathbb{F}_q\mathrm{P}^n$ is then given with respect to the (bi-)quadric field Q' which is just translated with respect to H_∞ from Q at each point $p \in \mathbb{F}_q\mathrm{P}^n$ by the vector $t(p) = \frac{1}{2}a(p) := \frac{1}{2}m^{-1}F(p)$. $a(x(\tau)) \in \mathbb{F}_q^n$ is called **acceleration** at $x(\tau)$.

Remark 3.1.7.

1. Even though this definition is done for arbitrary dimensions, it can be shown that a four-dimensional spacetime suffices since all extra dimension are 'absorbed' by the light cone of this theory which makes them unmeasurable. This is due to the non-existence of 'Eikörpern' in higher dimensions. This should be noted, but, in general, there is no direct restriction on the dimension of the spacetime.

2. The hopping process indicated by the trajectory being a function of the integers $\mathbb{Z}$ should be seen as the finite analogue of a continuous trajectory. We will also refer to a point $x(\tau)$ of a trajectory $x\colon \mathbb{Z} \to \mathbb{F}_q\mathrm{P}^n$ as x_τ with the index as a label of the step in this process.

3. Note that forward and backward velocity are defined between two neighbouring points whereas the mean velocity is defined at the event itself. Yet, even for the definition of the mean velocity, a local affine chart is needed.

4. The last point of this definition encapsulates a Finite Projective Physics analogue of Newton's second law, i. e. $F = ma$. It should be noted that the translation due to a force field with respect to some hyperplane should also be taken into account when considering a suitable hyperplane at infinity for the velocities at a point. The factor $\frac{1}{2}$ takes into account the fact that this situation describes the change due to two hoppings, from the predecessor to the event itself and from the event to its successor.

5. Up to now, no restrictions on the force field are enforced. Suitable force fields should preserve the causal structure of our spacetime, i. e. the tuple $(\mathbb{F}_q\mathrm{P}^n, Q')$ should also be a spacetime with a causal (bi-)quadric field.

6. Note that in order to mimic usual mechanics and to define a global force field, the motion of the particle seems to be constraint to an affine subspace. An involvement of the hyperplane at infinity could be a starting point for quantum theoretic or gravitational models on this spacetime. However, this has not been tried yet and is beyond the scope of this thesis.

One of the first questions we may ask ourselves is how the different velocities change in the presence of a force field. Thus, we will consider a flat and force-free spacetime in the sense that all bi-quadrics are just translated versions of the standard Minkowski

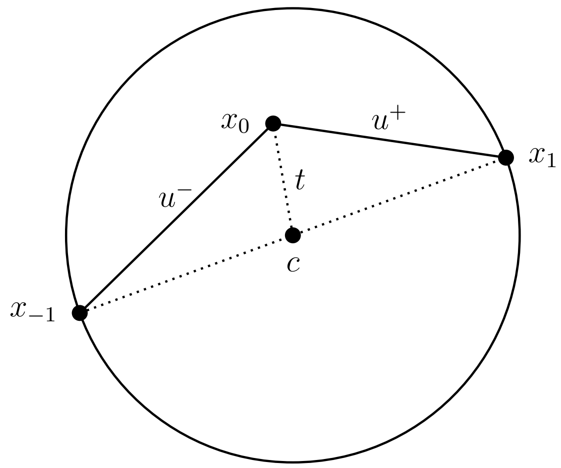


Figure 3.1: Illustration of a quadric with centre c located at x_0 as a circle around c in an affine subspace. x_{-1} denotes the predecessor of x_0, x_1 its successor. u^- and u^+ denote the translation vectors from x_{-1} to x_0 and from x_0 to x_1, respectively. The quadric centre is shifted from its location by the vector t. Nevertheless, the successor event x_1 lies on the line connecting the centre c of the quadric and the predecessor x_{-1}. Both, x_{-1} and x_1, lie on the quadric located at x_0.

bi-quadric which is located at $c = [0 : \ldots : 0 : 1]$. We will choose a global hyperplane at infinity $H_\infty = \{x \in \mathbb{F}_q\mathrm{P}^n \mid x_n = 0\}$ such that translations with respect to H_∞ are globally defined.

The bi-quadric field in the affine subspace $G_n = \mathbb{F}_q\mathrm{P}^n \setminus H_\infty$ will then be given by the representation matrices

$$M^\pm(p) = (T_{c\to p}^{-1})^T \eta^\pm T_{c\to p}^{-1} \tag{3.3}$$

at $p \in G_n$ where $T_{c\to P} \in \mathrm{PGL}(n+1, \mathbb{F}_q)$ is the representation matrix of the projectivity induced by a translation from c to p in the affine subspace G_n.

If an additional force field $F\colon \mathbb{F}_q\mathrm{P}^n \to \mathbb{F}_q^n$ is applied to a body with mass $m \in \mathbb{F}_q^\times$, the field of representation matrices in the affine subspace G_n at $p \in \mathbb{F}_q\mathrm{P}^n$ is then given by

$$M'^\pm(p) = (T_{(2m)^{-1}F(p)}^{-1})^T M^\pm(p) T_{(2m)^{-1}F(p)}^{-1} = (T_{\frac{1}{2}a(p)}^{-1})^T M^\pm(p) T_{\frac{1}{2}a(p)}^{-1} \tag{3.4}$$

with the acceleration $a(p) = m^{-1}F(p)$.

This additional translation by the acceleration can be visualized as shown in Figure 3.1.

To conceptualize this transformational behaviour of the representation matrices of the (bi-)quadrics, we want to introduce an action of the group $\mathrm{GL}(n+1, \mathbb{K})$ for some field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) \neq 2$ onto the set of homogeneous polynomials of degree two with coefficients in $\mathbb{K}$ in $n+1$ variables. This action should represent the transformational behaviour of the representation matrices of these homogeneous polynomials under some element of $\mathrm{PGL}(n+1, \mathbb{K})$.

If we consider a homogeneous polynomial $p \in \mathbb{K}[X_0, \dots, X_n]^{\text{hom},2} \subseteq \mathbb{K}[X_0, \dots, X_n]$ of degree two in $n+1$ variables, it can be represented by a (possibly symmetric) matrix $M_p \in \text{Mat}((n+1) \times (n+1), \mathbb{K})$ such that

$$p(x) = \sum_{i,j=0}^{n} a_{ij} x_i x_j \stackrel{!}{=} x^T M_p x$$

with $x = (x_0, \dots, x_m)^T \in \mathbb{K}^{n+1}$.

Usually, a coordinate transformation $x \mapsto x' = Sx$ with $S \in \text{GL}(n+1, \mathbb{K})$ would lead to a transformation of the form $M_p \mapsto M_p' = (S^{-1})^T M_p S^{-1}$ due to the condition

$$p(x) = x^T M_p x \stackrel{!}{=} x'^T M_p' x' = x^T S^T M_p' S x = p'(x').$$

Now, we want to consider only the change of the representation matrix and evaluate both in the same point, i. e. consider

$$p'(x) = x^T M_p' x = x^T (S^{-1})^T M_p S^{-1} x = p(S^{-1}x).$$

This will, again, be a homogeneous polynomial of degree two in $n+1$ variables.

This leads us to the definition of our action $\triangleright \colon \text{GL}(n+1, \mathbb{K}) \times \mathbb{K}[X_0, \dots, X_n]^{\text{hom},2} \to \mathbb{K}[X_0, \dots, X_n]^{\text{hom},2}, (S, p) \mapsto S \triangleright p$ with $(S \triangleright p)(x) = p(S^{-1}x)$.

Lemma 3.1.8. Let $\mathbb{K}$ be a field with $\text{char}(\mathbb{K}) \neq 2$ and let $\mathbb{K}[X_0, \dots, X_n]^{\text{hom},2}$ denote the set of homogeneous polynomials in $n+1 \in \mathbb{N}$ variables of degree two.

The map $\triangleright \colon \text{GL}(n+1, \mathbb{K}) \times \mathbb{K}[X_0, \dots, X_n]^{\text{hom},2} \to \mathbb{K}[X_0, \dots, X_n]^{\text{hom},2}, (S, p) \mapsto S \triangleright p$ with $(S \triangleright p)(x) = p(S^{-1}x)$ defines an action of $\text{GL}(n+1, \mathbb{K})$ onto $\mathbb{K}[X_0, \dots, X_n]^{\text{hom},2}$.

Proof. We have already seen that $S \triangleright p$ for $S \in \text{GL}(n+1, \mathbb{K})$ and $p \in \mathbb{K}[X_0, \dots, X_n]^{\text{hom},2}$ is again a homogeneous polynomial of degree two by the use of representation matrices. Thus, we only have to show that for $S, T \in \text{GL}(n+1, \mathbb{K})$ we have $S \triangleright (T \triangleright p) = (ST) \triangleright p$.

We immediately find that $(S \triangleright (T \triangleright p))(x) = (T \triangleright p)(S^{-1}x) = p(T^{-1}S^{-1}x) = p((ST)^{-1}x) = ((ST) \triangleright p)(x)$. □

Remark 3.1.9. Since we are only interested in the zero locus of a homogeneous polynomial of degree two, we can identify homogeneous polynomials which only differ by a non-zero scalar multiple, i. e. $p \sim q \iff q = \lambda p$ with $\lambda \in \mathbb{K}^\times$. Thus, we can 'projectivize' this action in the usual sense and only consider polynomials and matrices up to a non-zero scalar multiple. This leads to a projective version of this lemma. Hence, this can also be applied in the projective setting.

This directly leads to an action of $\text{PGL}(n+1, \mathbb{K})$ onto the set Q^n_{nd} of non-degenerate quadrics on $\mathbb{K}\text{P}^n$ by

$$S \triangleright Q := \{[x] \in \mathbb{K}\text{P}^n \mid (S \triangleright p)(x) = 0\} \tag{3.5}$$

with $S \in \text{PGL}(n+1, \mathbb{K}), Q = \{[x] \in \mathbb{K}\text{P}^n \mid p(x) = 0\} \in \text{Q}^n_{nd}$ and $p \in \mathbb{K}[X_0, \dots, X_n]^{\text{hom},2}$. This is projectively well-defined since the action $S \triangleright p$ as shown above leads to another homogeneous polynomial of degree two which is non-degenerate if and only if p is non-degenerate since S is invertible.

This can also be considered as an action onto the set BQ^n_{nd} of non-degenerate bi-quadrics on $\mathbb{K}\mathrm{P}^n$ by $S \triangleright (Q_1, Q_2) = (S \triangleright Q_1, S \triangleright Q_2)$ for a bi-quadric $(Q_1, Q_2) \in \mathrm{BQ}^n_{nd}$. One would have to check whether this still ensures two intersections between every line through the centre of this bi-quadric and the bi-quadric such that $S \triangleright (Q_1, Q_2)$ is still a bi-quadric. This seems to be reasonable but has not been proven yet.

Remark 3.1.10. Note that this action can also be used to find transformations which do not change the polynomial or the corresponding quadric, i. e. matrices $S \in (\mathrm{P})\mathrm{GL}(n+1, \mathbb{K})$ such that $S \triangleright p = p$ for $p \in \mathbb{K}[X_0, \ldots, X_n]^{\mathrm{hom},2}$. This is just the stabilizer $(\mathrm{P})\mathrm{GL}(n+1, \mathbb{K})_p$ of the polynomial p. In the case of the standard Minkowski forms $\eta^\pm$, these transformations are called **Lorentz transformations** if they also leave the Hyperplane $H_\infty = \{x \in \mathbb{K}\mathrm{P}^n \mid x_n = 0\}$ invariant.

The stabilizer of a quadric-field could be regarded as a field of the stabilizers at each point such that at each point this structure gives the stabilizer of the quadric at this point, but this has not been considered properly yet and is beyond the scope of this thesis.

We can use this action to formalize the (bi-)quadric field in a finite flat and force-free spacetime as shown in (3.3). At some point $x \in G_n$ in the affine patch $G_n \subseteq \mathbb{F}_q\mathrm{P}^n$, the (bi-)quadric is given by

$$Q(p) = (Q(p)^+, Q(p)^-) = T_{c\to p} \triangleright (H^+, H^-) =: T_{c\to p} \triangleright H \tag{3.6}$$

with $H^\pm$ the quadrics corresponding to the standard Minkowski forms $\eta^\pm$ which are located at $c = [0 : \ldots : 0 : 1] \in G_n$. This allows us to work with these quadrics more efficiently by considering mostly their transformation matrices from the standard Minkowski forms.

Equation (3.4) can be considered in the same manner as

$$T_{\frac{1}{2}a(p)} \triangleright Q(p) = (T_{\frac{1}{2}a(p)} T_{c\to p}) \triangleright H = T_{\frac{1}{2}a(p)+w(p)} \triangleright H \tag{3.7}$$

at $p \in G_n$ with the translation vector $w(p) \in \mathbb{F}_q^n$ from c to p with respect to H_∞, the acceleration $a(p) = m^{-1}F(p)$, the mass $m \in \mathbb{F}_q^\times$ and the force $F(p)$ at p where we have used the property of affine translations that successive translations can be viewed as one translation with the addition of the translation vectors of the different translations as its translation vector.

With this machinery, we can now answer the question of how the velocity vectors change when a force field is applied at a point $p \in G_n$.

Proposition 3.1.11. Let $\mathbb{F}_q$ be a finite field of odd order $q \in \mathbb{N}$ and let $(\mathbb{F}_q\mathrm{P}^n, Q)$ be a flat n-dimensional spacetime, $n \in \mathbb{N}$, which is given by the action of a force field $F\colon \mathbb{F}_q\mathrm{P}^n \to \mathbb{F}_q^n$ on a flat and force-free n-dimensional spacetime with respect to $H_\infty = \{x \in \mathbb{F}_q\mathrm{P}^n \mid x_n = 0\} \subseteq \mathbb{F}_q\mathrm{P}^n$, the hyperplane at infinity. Let $G_n = \mathbb{F}_q\mathrm{P}^n \setminus H_\infty$ denote the affine patch of the spacetime with respect to H_∞.

For a body with mass $m \in \mathbb{F}_q^\times$ at $p \in G_n$ with mean velocity $u_0 \in \mathbb{F}_q^n$, the backward velocity u^- and the forward velocity u^+ with respect to H_∞ at p are given, respectively, by

$$u^\pm = u_0 \pm \frac{1}{2}a(p) = u_0 \pm \frac{1}{2}m^{-1}F(p). \tag{3.8}$$

Proof. We start with a point $p = x_0 \in G_n$ as part of a trajectory of a body with mass $m \in \mathbb{F}_q^\times$. Its predecessor is denoted $x_{-1} \in G_n$ and is given by

$$x_0 = T_{u^-}x_{-1} = x_{-1} +_n u^-$$

where in the last step we defined informally the translation as a kind of addition with respect to H_∞. The successor of x_0 will be $x_1 \in G_n$ and be given by

$$x_1 = T_{u^+}x_0 = x_0 +_n u^+.$$

Both x_{-1} and x_1 are elements of the quadric $Q(p)^+ = Q(x_0)^+$, i. e. time-like neighbours of x_0.

The centre $z \in G_n$ of the quadric $Q(p)^+ = Q(x_0)^+$ is given by

$$z = T_{\frac{1}{2}a(p)}x_0 = x_0 +_n \frac{1}{2}a(p).$$

Thus, the laws of our finite dynamics state that x_1 should lie on the intersection between the quadric $Q(p)$ and the line connecting z and x_{-1}. Since x_{-1} and z lie in the affine patch G_n, we can find a translation vector $w \in \mathbb{F}_q^n$ such that $z = T_w x_{-1} = x_{-1} +_n w$. This line will be given by $\{T_{\lambda w}x_{-1} = x_{-1} +_n (\lambda w) \mid \lambda \in \mathbb{F}_q\} \subseteq G_n$ in the affine subspace. Thus, we need to find $\lambda \neq 0$ such that

$$x_1 = T_{\lambda w}x_{-1} \in Q(x_0)^+.$$

At first, we find due to the axioms for an affine space that

$$z = x_0 +_n \frac{1}{2}a(p) = x_{-1} +_n \left(u^- + \frac{1}{2}a(p)\right).$$

Additionally, we have already seen that $z = x_{-1} +_n w$. Thus, by the uniqueness of the translation vector in an affine space, we find that

$$w = u^- + \frac{1}{2}a(p).$$

This means that

$$x_1 = T_{\lambda(u^- + \frac{1}{2}a(p))}x_{-1} = T_{\lambda u^-}T_{\lambda\frac{1}{2}a(p)}T_{-u^-}x_0.$$

The polynomial representing the quadric $Q(x_0)^+$ is given by

$$q_0(x) = \left(T_{\frac{1}{2}a(p)}T_{c\to x_0} \rhd \eta^+\right)(x)$$

with $c = [0 : \dots : 0 : 1] \in G_n$ where by abuse of notation η^+ also denotes the homogeneous polynomial of degree two corresponding to the standard Minkowski form η^+.

Hence, we find with the additional property that translations with respect to the same hyperplane commute

$$\begin{aligned} q_0(x_1) &= \left(\left(T_{\frac{1}{2}a(p)} T_{c\to x_0}\right) \triangleright \eta^+\right)(x_1) \\ &= \left(\left(T_{\frac{1}{2}a(p)} T_{c\to x_0}\right) \triangleright \eta^+\right)\left(T_{\lambda u^-} T_{\lambda \frac{1}{2}a(p)} T_{-u^-} x_0\right) \\ &= \eta^+\left(T_{(\lambda-1)u^-} T_{\lambda\frac{1}{2}a(p)} T_{-\frac{1}{2}a(p)} T_{x_0\to c} x_0\right) \\ &= \eta^+\left(T_{(\lambda-1)\left(u^- + \frac{1}{2}a(p)\right)} c\right) \\ &= \eta^+\left(\begin{pmatrix} (\lambda-1)\left(u^- + \frac{1}{2}a(p)\right) \\ 1 \end{pmatrix}\right) \\ &= (\lambda-1)^2 \eta\left(u^- + \frac{1}{2}a(p)\right) + 1 \\ &= \rho\lambda^2 - 2\rho\lambda + \rho + 1 \overset{!}{=} 0 \end{aligned}$$

with the definition $\rho := \eta(u^- + \frac{1}{2}a(p))$ and the shorthand notation $\eta(x) = x^T \eta x$ for $x \in \mathbb{F}_q^n$.

We can solve this equation for λ by

$$\lambda_{1/2} = \frac{2\rho \pm \sqrt{4\rho^2 - 4\rho(\rho+1)}}{2\rho} = 1 \pm \frac{\sqrt{-\rho}}{\rho}.$$

To find a proper solution, we consider the condition $x_{-1} = T_{-u^-} x_0 \in Q(x_0)^+$. This means that

$$\begin{aligned} 0 = q_0(x_{-1}) &= \left(\left(T_{\frac{1}{2}a(p)} T_{c\to x_0}\right) \triangleright \eta^+\right)(x_{-1}) \\ &= \eta^+\left(T_{-\frac{1}{2}a(p)} T_{-u^-} T_{x_0\to c} x_0\right) \\ &= \eta^+\left(T_{-\frac{1}{2}a(p)} T_{-u^-} c\right) \\ &= \eta^+\left(\begin{pmatrix} -u^- - \frac{1}{2}a(p) \\ 1 \end{pmatrix}\right) \\ &= \eta\left(-u^- - \frac{1}{2}a(p)\right) + 1 = \eta\left(u^- + \frac{1}{2}a(p)\right) + 1. \end{aligned}$$

Thus, we found that $\rho = \eta(u^- + \frac{1}{2}a(p)) = -1$. Therefore,

$$\lambda_{1/2} = 1 \pm \frac{\sqrt{-\rho}}{\rho} = 1 \mp 1 \in \{0, 2\}.$$

The case $\lambda = 0$ would result in $x_1 = x_{-1}$ which we will disregard. The other solution $\lambda = 2$ leads to

$$x_1 = T_{2(u^- + \frac{1}{2}a(p))} x_{-1} = T_{u^- + a(p)} x_0 \overset{!}{=} T_{u^+} x_0.$$

With that, we found $u^+ = u^- + a(p)$. With $u_0 = \frac{u^+ + u^-}{2}$, we find $u_0 = \frac{2u^- + a(p)}{2} = u^- + \frac{1}{2}a(p)$ which yields at $p \in G_n$

$$u^- = u_0 - \frac{1}{2}a(p) \text{ and } u^+ = u_0 + \frac{1}{2}a(p).$$

□

Remark 3.1.12.

1. Note that here we have used the same hyperplane at infinity for all different points for the definition of the force field and for the definition of the translations for the different velocities. Note also that this result holds at a point $p \in G_n$ and has to be adapted at a different point by using the acceleration and mean velocity at that point.

2. The mean velocity $u_0(p)$ at some point $p \in G_n$ can be regarded as an initial condition for solving the equations of motion just as in the usual manner with a second order differential equation where we need an initial position and velocity for a unique solution.

Remark 3.1.13. The result from above is only valid in this form in a flat vacuum spacetime onto which a force field is applied which changes the (bi-)quadric field accordingly. But the methods shown above can be adapted to tackle even harder challenges and spacetimes if the intersection of the line from the predecessor through the centre of the quadric with the quadric lies in the affine subspace as specified before. Otherwise, this method via translations is not possible. The most noticeable difference of non-translational quadric fields would be that the transformation matrices of the polynomial to bring it into the standard Minkowski form may not commute with the ones of translations. Thus, one has to handle these steps with care.

Consider for example at a point $p \in \mathbb{F}_q\mathrm{P}^n$ the bi-quadric

$$Q(p) = (Q'(p)^+, Q'(p)^-) = \left(T_{\frac{1}{2}a(p)}M\right) \triangleright H = \left(T_{\frac{1}{2}a(p)}T_{c\to p}M'\right) \triangleright H$$

with $M, M' \in \mathrm{PGL}(n+1, \mathbb{F}_q)$, $M = T_{c\to p}M'$ and $c = [0 : \ldots : 0 : 1] \in G_n$. This means that we change the standard Minkowski quadrics by M' and then translate them to $T_{c\to p}$. This can always be formulated this way since $T_{c\to p}$ is invertible.

At first, we need to find the centre z of the bi-quadric $Q(p)$ which is done by using the symmetric representation matrix

$$M(Q(p)^+) = \left((T_{\frac{1}{2}a(p)}M)^{-1}\right)^T \eta^+ (T_{\frac{1}{2}a(p)}M)^{-1}$$

and multiplying its inverse with $h_\infty = (0, \ldots, 0, 1)^T \in \mathbb{F}_q^n$, i. e.

$$z = T_{\frac{1}{2}a(p)}M\eta^+M^T T^T_{\frac{1}{2}a(p)}h_\infty = T_{\frac{1}{2}a(p)}T_{c\to p}M'\eta^+M'^T h_\infty$$

since here $T_v^T h_\infty = h_\infty$ for any translation vector $v \in \mathbb{F}_q^n$.

If $z \in G_n$ is in the affine subspace, we can find the unique translation vector $w \in \mathbb{F}_q^n$ from x_{-1} to z, i.e. $z = T_w x_{-1}$. x_1 should then lie in the affine subspace on this line connecting z and x_{-1} in the affine subspace, i.e. $x_1 = T_{\lambda w} x_{-1}$ for some $\lambda \in \mathbb{F}_q$. If z lies in the hyperplane at infinity, we can find the line connecting z and x_{-1} by the set $\{\delta z + \mu x_{-1} \mid (\delta, \mu) \in \mathbb{F}_q^2 \setminus \{(0,0)\}\}$ which will lead to $w = \tilde{z}$ with $z = [\tilde{z} : 0]$ as translational vector on the line in the affine subspace. Note that since only non-zero scalar multiples of w contribute to the description of the line, this is well-defined in the projective sense.

With the conventions as above, we find that

$$q_0(x_1) = \left(\left(T_{\frac{1}{2}a(p)} M\right) \triangleright \eta^+\right)(x_1) = \left(M' \triangleright \eta^+\right)\left(T_{\lambda w - u^- - \frac{1}{2}a(p)} c\right)$$

which leads to

$$q_0(x_1) = \eta^+ \left(M'^{-1} \begin{pmatrix} \lambda w - u^- - \frac{1}{2}a(p) \\ 1 \end{pmatrix}\right) \stackrel{!}{=} 0.$$

The condition

$$q_0(x_{-1}) = \eta^+ \left(M'^{-1} \begin{pmatrix} -u^- - \frac{1}{2}a(p) \\ 1 \end{pmatrix}\right) = 0$$

can be used to reduce the above equation as shown earlier. This will lead to a quadratic equation in λ with one solution being $\lambda_1 = 0$, i.e. $x_1 = x_{-1}$ which would mimic a time reversal at x_0. The other solution λ_2, if it exists which should be the case if the transformation M is chosen properly, yields the successor x_1 of x_0 by the described principles. Identifying the translation vectors with respect to G_n from x_{-1} to x_0 with the backward velocity u^- and the translation vector from x_0 to x_1 with the forward velocity u^+ by means of

$$x_1 = T_{\lambda_2 w} x_{-1} = T_{\lambda_2 w - u^-} x_0 \stackrel{!}{=} T_{u^+} x_0$$

will give us the kinematical description of this system, i.e. $u^+ = \lambda_2 w - u^-$ such that

$$u_0 = \frac{u^+ + u^-}{2} = \frac{\lambda_2}{2} w$$

where λ_2 may also depend on u^- and $a(p)$. This is heavily influenced by the structure of the quadric at x_0 which is to be expected since all information on the physical dynamics in this spacetime is encoded in the (bi-)quadric field.

Using these results, we can now describe the trajectory of a body with mass $m \in \mathbb{F}_q^\times$ with initial mean velocity $u_0(p) \in \mathbb{F}_q^n$ at a point $p \in G_n$.

Proposition 3.1.14. Let $\mathbb{F}_q$ be a finite field of odd order $q \in \mathbb{N}$ and let $(\mathbb{F}_q\mathrm{P}^n, Q)$ be a flat n-dimensional spacetime which is shifted from a vacuum spacetime by translations corresponding to a force field $F\colon \mathbb{F}_q\mathrm{P}^n \to \mathbb{F}_q^n$ with respect to $H_\infty = \{x \in \mathbb{F}_q\mathrm{P}^n \mid x_n = 0\}$.

The trajectory $x\colon \mathbb{Z} \to \mathbb{F}_q\mathrm{P}^n$ of a body with mass $m \in \mathbb{F}_q^\times$ and initial velocity $u_0(p) \in \mathbb{F}_q^n$ at $p = x(0) \in G_n = \mathbb{F}_q\mathrm{P}^n \setminus H_\infty$ is given by

$$x(\tau) = x_\tau = \begin{cases} x_0 +_n \left(\sum_{i=0}^{\tau-1}\left(u_0(x_i) + \frac{1}{2}m^{-1}F(x_i)\right)\right), & \text{for } \tau > 0, \\ x_0 +_n \left(\sum_{i=\tau+1}^{0}\left(-u_0(x_i) + \frac{1}{2}m^{-1}F(x_i)\right)\right), & \text{for } \tau < 0. \end{cases} \tag{3.9}$$

Proof. The proof is rather straight forward. For $\tau > 0$, we see that

$$x(\tau) = x_\tau = x_{\tau-1} +_n u^+(x_{\tau-1}) = \ldots = x_0 +_n \left(u^+(x_0) + u^+(x_1) + \ldots + u^+(x_{\tau-1})\right).$$

Inserting $u^+(p) = u_0(p) + \frac{1}{2}m^{-1}F(p)$ for $p \in G_n$ from (3.8), we immediately find

$$x_\tau = x_0 +_n \left(\sum_{i=0}^{\tau-1} \left(u_0(x_i) + \frac{1}{2}m^{-1}F(x_i)\right)\right).$$

For $\tau < 0$, the procedure is analogous. We see that by definition $x_\tau = x_{\tau+1} +_n (-u^-(x_{\tau+1}))$. Iteratively, this yields

$$x_\tau = x_0 +_n (-u^-(x_0) - u^-(x_1) - \ldots - u^-(x_{\tau+1})).$$

Again, inserting $u^-(p) = u_0(p) - \frac{1}{2}m^{-1}F(p)$ from (3.8) gives us

$$x_\tau = x_0 +_n \left(\sum_{i=\tau+1}^{0} \left(-u_0(x_i) + \frac{1}{2}m^{-1}F(x_i)\right)\right).$$

□

The only thing that is left to determine is the mean velocity at some point x_τ with respect to the initial mean velocity $u_0(x_0)$. These can be determined using the condition that $u^+(x_{\tau-1}) = u^-(x_\tau)$ in the flat case such that there is no 'hole' between these two points and their forward and backward velocities.

Proposition 3.1.15. Let the situation be as in 3.1.14.
The mean velocity at a point x_τ, $\tau \in \mathbb{Z} \setminus \{0\}$, is given by

$$u_0(x_\tau) = \begin{cases} u_0(x_0) + \frac{1}{2}m^{-1}\left(F(x_0) + 2\sum_{i=1}^{\tau-1} F(x_i) + F(x_\tau)\right), & \text{for } \tau > 0, \\ u_0(x_0) + \frac{1}{2}m^{-1}\left(F(x_0) + 2\sum_{i=\tau+1}^{-1} F(x_i) + F(x_\tau)\right), & \text{for } \tau < 0. \end{cases} \tag{3.10}$$

Proof. Let us consider for $\tau > 0$ the point x_1 and its mean velocity $u_0(x_1)$. The condition $u^+(x_{\tau-1}) = u^-(x_\tau)$ together with (3.8) tells us that

$$u_0(x_0) + \frac{1}{2}m^{-1}F(x_0) = u^+(x_0) \overset{!}{=} u^-(x_1) = u_0(x_1) - \frac{1}{2}m^{-1}F(x_1).$$

This immediately yields

$$u_0(x_1) = u_0(x_0) + \frac{1}{2}m^{-1}\left(F(x_0) + F(x_1)\right).$$

Following the ansatz above, we iteratively find

$$u_0(x_2) = u_0(x_1) + \frac{1}{2}m^{-1}\left(F(x_1) + F(x_2)\right) = u_0(x_0) + \frac{1}{2}m^{-1}\left(F(x_0) + 2F(x_1) + F(x_2)\right)$$

and, generally, for $\tau \in \mathbb{N}$

$$u_0(x_\tau) = u_0(x_0) + \frac{1}{2}m^{-1}\left(F(x_0) + 2\sum_{i=1}^{\tau-1} F(x_i) + F(x_\tau)\right).$$

For $\tau < 0$, using $u_0(x_{-1}) + \frac{1}{2}m^{-1}F(x_{-1}) = u_0(x_0) - \frac{1}{2}m^{-1}F(x_0)$, we find

$$u_0(x_{-1}) = u_0(x_0) - \frac{1}{2}m^{-1}\left(F(x_0) + F(x_{-1})\right).$$

This yields inductively

$$u_0(x_\tau) = u_0(x_0) - \frac{1}{2}m^{-1}\left(F(x_0) + 2\sum_{i=\tau+1}^{-1} F(x_i) + F(x_\tau)\right).$$

□

Remark 3.1.16.

1. This representation may be useful if the forces $F(p)$ do not depend on the mean velocities u_0. Otherwise, it is more useful to separate terms depending on $u_0(x_0)$ and $u_0(x_1)$, respectively. This will become useful in the description of finite electrodynamics where the *Lorentz force* is given by $F_L(p) = \tilde{F}(p)u_0(p)$ with some Matrix $\tilde{F}(p) \in \mathrm{Mat}(4 \times 4, \mathbb{F}_q)$ representing the *Faraday tensor*.

2. Note that the nature of the forces $F(x_n)$ should be such that $u_0(x_\tau)^T \eta u_0(x_\tau) = -1$ for all $\tau \in \mathbb{Z}$.

These considerations now yield the trajectory of a massive body subject to a given force field. Note however that this theory does not yet give rise to any restrictions on the force fields besides that the new quadric field should remain causal.

3.2 Finite Projective Electrodynamics

As one of the first gauge theories, electrodynamics, or more advanced Quantum Electrodynamics, offers a rich and non-trivial playground to bring our theory to the test. It was also one of the starting points of the development of Special Relativity. Thus, since our theory is inherently relativistic, finite Newtonian Mechanics of a charged body in an electromagnetic field should lead to trajectories which should be in close comparison with an analytic solution in the usual relativistic mechanics picture.

Here, the force onto a charged body with mass $m \in \mathbb{R}^\times$ and charge $q \in \mathbb{R}$ is given by the *Lorentz four-force* $F_L \in \mathbb{R}^{1+3}$, in components and natural units

$$F_L^\mu = qF^\mu_\nu u^\nu$$

with the four-velocity $u \in \mathbb{R}^{1+3}$ and the *Faraday tensor*

$$F = (F_{\mu\nu}) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}.$$

$E_1, E_2, E_3, B_1, B_2, B_3 \in \mathbb{R}$ denote the components of the electric field $\vec{E} = (E_1, E_2, E_3)^T \in \mathbb{R}^3$ and of the magnetic field $\vec{B} = (B_1, B_2, B_3)^T \in \mathbb{R}^3$, respectively. These two field obey Maxwell's equations

$$\mathrm{d}F = \mathrm{d} \star F = 0$$

if all sources vanish, with the Hodge star $\star$.

Solutions of Newton's second law of motion $F_L = ma$ with the four-acceleration $a = \dot{u} = \ddot{x} \in \mathbb{R}^{1+3}$ lead to the trajectory $x(\tau)$ of a body with charge q and mass m.

At first we will inspect the motion of a charged and massive body in an electromagnetic field in the framework of Finite Newtonian Mechanics in more detail. Afterwards, we will compare the computation of such a trajectory in Finite Projective Electrodynamics with the one of an analytic solution using a static and homogeneous electromagnetic field for simplicity.

3.2.1 Finite Newtonian Mechanics of a Charged Body in an Electromagnetic field

Finite Projective Electrodynamics should resemble laws of motion with the Lorentz four-force as the source of the change in the trajectory of a body with mass $m \in \mathbb{F}_q^\times$ and charge $\tilde{q} \in \mathbb{F}_q$. We adapt the notions of Finite Newtonian Mechanics to this new situation and restrict ourselves to four dimensions.

Definition 3.2.1. Let $\mathbb{F}_q$ be a finite field of odd order $q \in \mathbb{N}$.

1. The **(finite) Faraday matrix field** $\tilde{F} \colon \mathbb{F}_q\mathrm{P}^4 \to \mathrm{Mat}(4\times 4, \mathbb{F}_q)$ is given at $p \in \mathbb{F}_q\mathrm{P}^4$ by

$$\tilde{F}(p) = (F^\mu_\nu(p)) = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \tag{3.11}$$

with the components $E_1, E_2, E_3 \in \mathbb{F}_q$ of the electric field at the point p and the components $B_1, B_2, B_3 \in \mathbb{F}_q$ of the magnetic field at p.

2. The **(finite) Lorentz force field** $F_L \colon \mathbb{F}_q\mathrm{P}^4 \to \mathbb{F}_q^4$ acting on a body with charge $\tilde{q} \in \mathbb{F}_q$ and mean velocity field u_0 is given by

$$F_L(p) = \tilde{q}\tilde{F}(p)u_0(p) \tag{3.12}$$

at a point $p \in \mathbb{F}_q\mathrm{P}^4$.

Remark 3.2.2. It is important that the Lorentz force depends on the mean velocity since this velocity is located at a point and not between two points as in the case of the forward or backward velocity. There is also the possibility to define a forward and backward Faraday tensor such that

$$F_L = \tilde{q}\frac{F^+u^+ + F^-u^-}{2} \tag{3.13}$$

such that if $F^+ = F^- = F$ we find

$$\frac{F^+u^+ + F^-u^-}{2} = F\frac{u^+ + u^-}{2} = Fu_0.$$

This might be useful in a non-flat spacetime, but we will refer to the former definition as of now in a flat spacetime.

As a first step, we can use the condition $u^-(x_\tau) = x^+(x_{\tau-1})$ to find the mean velocity at a point $x(\tau)$ of the trajectory of a body of mass $m \in \mathbb{F}_q^\times$ and charge $\tilde{q} \in \mathbb{F}_q$ subject to the Lorentz force F_L with respect to some electromagnetic field encoded in the Faraday matrix field $\tilde{F}$. We will see that in the special case that $\tilde{F}(p) = \tilde{F}$ for all $p \in \mathbb{F}_q\mathrm{P}^n$ and a fixed matrix $\tilde{F}$, i. e. a static and homogeneous electromagnetic field, the transformation between the mean velocity at some point to the next point of the trajectory is given by a *Lorentz transformation*, i. e. a matrix $L \in \mathrm{Mat}(4 \times 4, \mathbb{F}_q)$ such that $L^T\eta L = \eta$.

Proposition 3.2.3. Let $\mathbb{F}_q$ be a finite field of odd order $q \in \mathbb{N}$ and let $(\mathbb{F}_q\mathrm{P}^4, Q)$ be a flat 4-dimensional spacetime which is shifted from a vacuum spacetime by translations corresponding to the Lorentz force field $F_L \colon \mathbb{F}_q\mathrm{P}^4 \to \mathbb{F}_q^4$ with respect to $H_\infty = \{x \in \mathbb{F}_q\mathrm{P}^4 \mid x_4 = 0\}$ which couples to a body with mass $m \in \mathbb{F}_q^\times$ and charge $\tilde{q} \in \mathbb{F}_q$. Let $x \colon \mathbb{Z} \to \mathbb{F}_q\mathrm{P}^4$ be the trajectory of this body with initial mean velocity $u_0(x_0) \in \mathbb{F}_q^4$ at $x_0 \in G_4 = \mathbb{F}_q\mathrm{P}^4 \setminus H_\infty$.

The mean velocity $u_0(x_\tau)$ for $\tau \in \mathbb{N}$ is given by

$$u_0(x_\tau) = C(\tilde{F}(x_\tau))C(\tilde{F}(x_{\tau-1}))\cdots C(\tilde{F}(x_1))u_0(x_0) \tag{3.14}$$

with

$$C(\tilde{F}(x_i)) = \left(I_4 - \frac{\tilde{q}}{2m}\tilde{F}(x_i)\right)^{-1}\left(I_4 + \frac{\tilde{q}}{2m}\tilde{F}(x_{i-1})\right) \tag{3.15}$$

for $i \in \{1, \ldots, \tau\}$ if $I_4 - \frac{\tilde{q}}{2m}\tilde{F}(x_i)$ is invertible where $I_4 \in \mathrm{Mat}(4 \times 4, \mathbb{F}_q)$ denotes the identity matrix. Analogous for $\tau < 0$ in the inverse direction.

If $\tilde{F}(p) = \tilde{F} \in \mathrm{Mat}(4\times 4, \mathbb{F}_q)$ for all $p \in \mathbb{F}_q\mathrm{P}^4$, i. e. the electromagnetic field is constant on the spacetime, we find that for any x_i we have

$$C(\tilde{F}) = \left(I_4 - \frac{\tilde{q}}{2m}\tilde{F}\right)^{-1}\left(I_4 + \frac{\tilde{q}}{2m}\tilde{F}\right) = \left(I_4 + \frac{\tilde{q}}{2m}\tilde{F}\right)\left(I_4 - \frac{\tilde{q}}{2m}\tilde{F}\right)^{-1} \tag{3.16}$$

and

$$C(\tilde{F})^T\eta C(\tilde{F}) = \eta, \tag{3.17}$$

i. e. $C(\tilde{F})$ is a Lorentz transformation, which means that $C \colon \mathfrak{o}((1,3), \mathbb{F}_q) \to O((1,3), \mathbb{F}_q)$ is a Cayley transform.

Proof. We start by inspecting the condition $u^-(x_i) = u^+(x_{i-1})$. Additionally, we use (3.8) adapted to this situation, meaning that

$$u^-(x_i) = u_0(x_i) - \frac{1}{2m}F_L(x_i) = u_0(x_i) - \frac{\tilde{q}}{2m}\tilde{F}(x_i)u_0(x_i) = \left(I_4 - \frac{\tilde{q}}{2m}\tilde{F}(x_i)\right)u_0(x_i)$$

and analogously

$$u^+(x_i) = u_0(x_i) + \frac{1}{2m}F_L(x_i) = \left(I_4 + \frac{\tilde{q}}{2m}\tilde{F}(x_i)\right)u_0(x_i).$$

With the condition above, this immediately yields

$$\left(I_4 - \frac{\tilde{q}}{2m}\tilde{F}(x_i)\right)u_0(x_i) = \left(I_4 + \frac{\tilde{q}}{2m}\tilde{F}(x_{i-1})\right)u_0(x_{i-1}).$$

Inserting $i = 1$ leads to the equation

$$\left(I_4 - \frac{\tilde{q}}{2m}\tilde{F}(x_1)\right)u_0(x_1) = \left(I_4 + \frac{\tilde{q}}{2m}\tilde{F}(x_0)\right)u_0(x_0).$$

If $\frac{\tilde{q}}{2m}$ is no eigenvalue of $\tilde{F}(x_1)$, $\left(I_4 - \frac{\tilde{q}}{2m}\tilde{F}(x_1)\right)$ is invertible. Thus,

$$u_0(x_1) = \left(I_4 - \frac{\tilde{q}}{2m}\tilde{F}(x_1)\right)^{-1}\left(I_4 + \frac{\tilde{q}}{2m}\tilde{F}(x_0)\right)u_0(x_0) = C(\tilde{F}(x_1))u_0(x_0).$$

An iteration of this process leads to

$$\begin{aligned} u_0(x_\tau) &= C(\tilde{F}(x_\tau))u_0(x_{\tau-1}) = C(\tilde{F}(x_\tau))C(\tilde{F}(x_{\tau-1}))u_0(x_{\tau-2}) = \ldots = \\ &= C(\tilde{F}(x_\tau))C(\tilde{F}(x_{\tau-1}))\ldots C(\tilde{F}(x_1))u_0(x_0) \end{aligned}$$

as desired.

If the Faraday matrix field is constant, i. e. at all points $p \in \mathbb{F}_q\mathrm{P}^4$ we have $\tilde{F}(p) = \tilde{F}$ for some fixed $\tilde{F} \in \mathrm{Mat}(4 \times 4, \mathbb{F}_q)$,

$$C(\tilde{F}(p)) = C(\tilde{F}) = \left(I_4 - \frac{\tilde{q}}{2m}\tilde{F}\right)^{-1}\left(I_4 + \frac{\tilde{q}}{2m}\tilde{F}\right).$$

A calculation in analogy to calculations regarding the Cayley transform from skew-symmetric matrices to orthogonal matrices shows that $\left(I_4 - \frac{\tilde{q}}{2m}\tilde{F}\right)^{-1}$ and $I_4 + \frac{\tilde{q}}{2m}\tilde{F}$ commute. Thus, we find that

$$C(\tilde{F}) = \left(I_4 - \frac{\tilde{q}}{2m}\tilde{F}\right)^{-1}\left(I_4 + \frac{\tilde{q}}{2m}\tilde{F}\right) = \left(I_4 + \frac{\tilde{q}}{2m}\tilde{F}\right)\left(I_4 - \frac{\tilde{q}}{2m}\tilde{F}\right)^{-1}.$$

We will make use of this in the following.

To confirm that $C(\tilde{F})$ is indeed a Lorentz transformation, we observe that $\left(\eta\tilde{F}\right)^T = -\eta\tilde{F}$ and $\left(\tilde{F}\eta\right)^T = -\tilde{F}\eta$ such that

$$\eta\tilde{F}^T\eta = -\tilde{F} \text{ and } \tilde{F}^T\eta + \eta\tilde{F} = 0$$

since $\eta^T = \eta^{-1} = \eta$. Now, we can evaluate $C(\tilde{F})^T \eta C(\tilde{F})$.

$$\begin{aligned}
C(\tilde{F})^T \eta C(\tilde{F}) &= \left(\left(I_4 - \frac{\tilde{q}}{2m}\tilde{F}\right)^{-1}\left(I_4 + \frac{\tilde{q}}{2m}\tilde{F}\right)\right)^T \eta \left(I_4 - \frac{\tilde{q}}{2m}\tilde{F}\right)^{-1}\left(I_4 + \frac{\tilde{q}}{2m}\tilde{F}\right) \\
&= \left(I_4 - \frac{\tilde{q}}{2m}\tilde{F}^T\right)^{-1}\left(I_4 + \frac{\tilde{q}}{2m}\tilde{F}^T\right)\eta\left(I_4 + \frac{\tilde{q}}{2m}\tilde{F}\right)\left(I_4 - \frac{\tilde{q}}{2m}\tilde{F}\right)^{-1} \\
&= \left(I_4 - \frac{\tilde{q}}{2m}\tilde{F}^T\right)^{-1}\left(I_4 + \frac{\tilde{q}}{2m}\left(\underbrace{\tilde{F}^T\eta + \eta\tilde{F}}_{=0}\right) + \frac{\tilde{q}^2}{4m^2}\tilde{F}^T\eta\tilde{F}\right)\left(I_4 - \frac{\tilde{q}}{2m}\tilde{F}\right)^{-1} \\
&= \left(I_4 - \frac{\tilde{q}}{2m}\tilde{F}^T\right)^{-1}\left(I_4 + \frac{\tilde{q}^2}{4m^2}\tilde{F}^T\eta\tilde{F}\right)\left(I_4 - \frac{\tilde{q}}{2m}\tilde{F}\right)^{-1} \\
&= \left(I_4 - \frac{\tilde{q}}{2m}\tilde{F}^T\right)^{-1}\left(I_4 - \frac{\tilde{q}}{2m}\left(\underbrace{\tilde{F}^T\eta + \eta\tilde{F}}_{=0}\right) + \frac{\tilde{q}^2}{4m^2}\tilde{F}^T\eta\tilde{F}\right)\left(I_4 - \frac{\tilde{q}}{2m}\tilde{F}\right)^{-1} \\
&= \left(I_4 - \frac{\tilde{q}}{2m}\tilde{F}^T\right)^{-1}\left(I_4 - \frac{\tilde{q}}{2m}\tilde{F}^T\right)\eta\left(I_4 - \frac{\tilde{q}}{2m}\tilde{F}\right)\left(I_4 - \frac{\tilde{q}}{2m}\tilde{F}\right)^{-1} \\
&= \eta.
\end{aligned}$$

Thus, $C(\tilde{F})$ is a Lorentz transformation, i.e. $C(\tilde{F}) \in O((1,3),\mathbb{F}_q)$. Because of the relation $\tilde{F}^T\eta + \eta\tilde{F} = 0$, we see that $\tilde{F} \in \mathfrak{o}((1,3),\mathbb{F}_q)$, i.e. it is an element of the Lie algebra $\mathfrak{o}((1,3),\mathbb{F}_q)$ of Lorentz transformations. Therefore, $C\colon \mathfrak{o}((1,3),\mathbb{F}_q) \to O((1,3),\mathbb{F}_q)$ is a Cayley transform. □

Remark 3.2.4. Note that $C\left(\tilde{F}(x_i)\right)$ is a mediator between the mean velocities $u_0(x_i)$ and $u_0(x_{i-1})$. Since the mean velocity has to fulfil $u_0^T\eta u_0 = -1$, this means that $C\left(\tilde{F}(x_i)\right)$ should be a Lorentz transformation to preserve this (pseudo-)norm. We will later see that, in the general case, this leads to certain quantities being constant along the trajectory which may be unphysical. Note however that according to Einstein's General Relativity the presence of an electromagnetic field changes the geometry which has not been taken into account so far.

Remark 3.2.5. It should not come as a surprise that the form of the Lorentz force leads to a Lorentz transformation mapping one mean velocity to another. Even in the usual description of mechanics, this is immediately implied since the Faraday matrix $\tilde{F}$ is an element of the Lorentz algebra $\mathfrak{o}(1,3)$ and is also a linear mediator between the velocity and the change thereof, i.e. acceleration, by Newton's laws of motion, i.e. $F = ma = m\dot{u} = q\tilde{F}u$. If $\tilde{F}$ is independent of proper time τ, this can be solved by

$$u(\tau) = \exp\left(\tau\frac{q}{m}\tilde{F}\right)u(0) = G(\tau)u(0)$$

which shows by means of Lie theory that $G(\tau)$ is a Lorentz transformation.

On the other hand, following the ideas of [1], consider a force law of the form $ma = Fu$ and a linear time evolution such that $u(\tau) = G(\tau)u(0)$. Since the pseudo-norm of u has to be constant, we immediately conclude that $G(\tau) \in \mathrm{O}(1,3)$. If we now take the derivative of this time evolution, we arrive at

$$\dot{u}(\tau) = \dot{G}(\tau)u(0) = \dot{G}(\tau)G(\tau)^{-1}u(\tau) =: F(\tau)u(\tau).$$

Using general results from Lie theory, we can see that $F(\tau)$ is an element of the Lorentz algebra. Obviously, this is of the form of the Lorentz force which depends linearly on the four-velocity u. This shows that that the linear mediator between u and the Lorentz force is an element of the Lorentz algebra even without knowing its internal structure.

Knowing the mean velocity of a charged body in a given electromagnetic field, we can now make use of (3.9) and compute the trajectory of this body due to the Lorentz force.

Corollary 3.2.6. Let $\mathbb{F}_q$ be a finite field of odd order $q \in \mathbb{N}$ and let $(\mathbb{F}_q\mathrm{P}^4, Q)$ be a flat 4-dimensional spacetime which is shifted from a vacuum spacetime by translations corresponding to the Lorentz force field $F_L\colon \mathbb{F}_q\mathrm{P}^4 \to \mathbb{F}_q^4$ with respect to $H_\infty = \{x \in \mathbb{F}_q\mathrm{P}^4 \mid x_4 = 0\}$ which couples to a body with mass $m \in \mathbb{F}_q^\times$ and charge $\tilde{q} \in \mathbb{F}_q$. Let $x\colon \mathbb{Z} \to \mathbb{F}_q\mathrm{P}^4$ be the trajectory of this body with initial mean velocity $u_0(x_0) \in \mathbb{F}_q^4$ at $x_0 \in G_4 = \mathbb{F}_q\mathrm{P}^4 \setminus H_\infty$.

The trajectory x at proper time $\tau \in \mathbb{N}$ is given by

$$x_\tau = x_0 +_4 \left(\sum_{i=0}^{\tau-1} \left(I_4 + \frac{\tilde{q}}{2m}\tilde{F}(x_i) \right) C(\tilde{F}(x_i))C(\tilde{F}(x_{i-1})) \cdots C(\tilde{F}(x_1))u_0(x_0) \right) \tag{3.18}$$

In particular, for a constant Faraday matrix field $\tilde{F}(p) = \tilde{F} \in \mathrm{Mat}(4 \times 4, \mathbb{F}_q)$ for all $p \in \mathbb{F}_q\mathrm{P}^4$ this trajectory at proper time $\tau \in \mathbb{N}$ is given by

$$x_\tau = x_0 +_4 \left(\left(I_4 + \frac{\tilde{q}}{2m}\tilde{F} \right) \left(\sum_{i=0}^{\tau-1} C(\tilde{F})^i \right) u_0(x_0) \right) \tag{3.19}$$

Proof. Inserting the special form of the Lorentz force into (3.9) for $\tau > 0$ and $n = 4$ leads to

$$x_\tau = x_0 +_4 \left(\sum_{i=0}^{\tau-1} \left(u_0(x_i) + \frac{\tilde{q}}{2m}\tilde{F}(x_i)u_0(x_i) \right) \right) = x_0 +_4 \left(\sum_{i=0}^{\tau-1} \left(I_4 + \frac{\tilde{q}}{2m}\tilde{F}(x_i) \right) u_0(x_i) \right).$$

Using (3.14) for $u_0(x_i) = C(\tilde{F}(x_i))C(\tilde{F}(x_{i-1})) \cdots C(\tilde{F}(x_1))u_0(x_0)$ yields the desired result

$$x_\tau = x_0 +_4 \left(\sum_{i=0}^{\tau-1} \left(I_4 + \frac{\tilde{q}}{2m}\tilde{F}(x_i) \right) C(\tilde{F}(x_i))C(\tilde{F}(x_{i-1})) \cdots C(\tilde{F}(x_1))u_0(x_0) \right).$$

For a constant Faraday matrix field $\tilde{F}(p) = \tilde{F} \in \mathrm{Mat}(4 \times 4, \mathbb{F}_q)$ for all $p \in \mathbb{F}_q\mathrm{P}^4$, we immediately see that

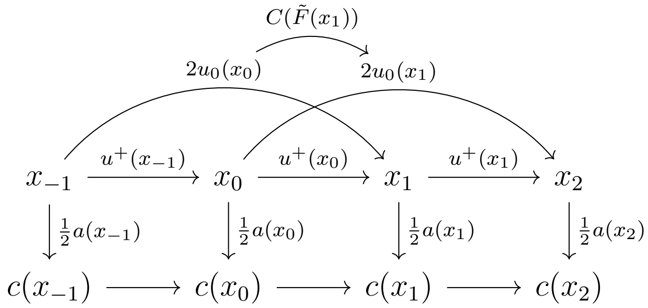
$$u_0(x_i) = C(\tilde{F})^i u_0(x_0).$$

Thus, we find after some re-bracketing

$$x_\tau = x_0 +_4 \left(\left(I_4 + \frac{\tilde{q}}{2m}\tilde{F} \right) \left(\sum_{i=0}^{\tau-1} C(\tilde{F})^i \right) u_0(x_0) \right)$$

for $\tau > 0$. □

Remark 3.2.7. The different mappings and translations on the different geometrical entities on a trajectory in an electromagnetic field can be visualized as follows with the usual notational conventions:



(3.20)

Following the paths of this diagram, the translation vectors between the centres $c(x_i)$ of the quadrics $Q(x_i)$ can be determined. Note that $C(\tilde{F}(x_1))$ acts by multiplication in contrast to the other arrows which indicate a translation.

Herewith, we can now compute the trajectory of a body with mass $m \in \mathbb{F}_q^\times$ with charge $\tilde{q} \in \mathbb{F}_q$ subject to an electromagnetic field.

3.2.2 Comparison with the Relativistic Analytical Solution in a Static and Homogeneous Electromagnetic Field

In order to check whether our ansatz provides a suitable description of the relativistic motion of a charged particle in an electromagnetic field, we will simplify our assumptions. We will work with a static and homogeneous electromagnetic field which means that the Faraday matrix is constant on the whole spacetime (or at least on the affine subspace), i. e. $\tilde{F}(p) = \tilde{F} \in \mathrm{Mat}(4 \times 4, \mathbb{F}_q)$ for all $p \in \mathbb{F}_q\mathrm{P}^4$.

For the analogue in usual relativistic mechanics, [6] provides analytical solutions of the trajectory of such a charged body and gives approximate solutions for special cases where certain combinations of the strength of the electric and magnetic field vanish. They use the usual framework of special relativistic electrodynamics and use an external electromagnetic field which is constant in spacetime.

We will provide a short overview of their findings. The particle of mass $m \in \mathbb{R}_+$ and charge $q \in \mathbb{R}$ has initial four-velocity V^μ and initial position P^μ. The electromagnetic field's components are encoded in the Faraday matrix

$$F = (F^\mu{}_\nu) = \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

and its Hodge-dual

$$G = (G^\mu{}_\nu) = \begin{pmatrix} 0 & B_x & B_y & B_z \\ B_x & 0 & -E_z & E_y \\ B_y & E_z & 0 & -E_x \\ B_z & -E_y & E_x & 0 \end{pmatrix}.$$

A re-definition of proper time τ as $\zeta = \frac{q\tau}{m}$ and a power series ansatz for $\dot{x} = \partial_\zeta x = \left(\sum_{n=0}^{\infty} \frac{1}{n!} (\zeta F)^n\right) \frac{mV}{q}$ leads to the definition of two parameters:

$$F^2 = G^2 + k_1 I_4 \text{ with } k_1 = ||E||^2 - ||B||^2 = \frac{1}{2}\mathrm{Tr}(F^2)$$

and

$$FG = GF = \frac{1}{2}k_2 I_4 \text{ with } k_2 = 2\vec{E}\cdot\vec{B} = \frac{1}{2}\mathrm{Tr}(FG).$$

With these, we can define two new constants:

$$a := \sqrt{\frac{\sqrt{k_1^2 + k_2^2} + k_1}{2}}$$

and

$$b := \sqrt{\frac{\sqrt{k_1^2 + k_2^2} - k_1}{2}}.$$

The solution of the trajectory of the particle with respect to ζ is then given as matrix-vector-multiplication if $k_2 \neq 0$:

$$\begin{aligned} x =& \frac{m}{q\left(a^2+b^2\right)} \left[\left(\frac{b^2\sinh(a\zeta)}{a} + \frac{a^2\sin(b\zeta)}{b}\right) I_4 \right. \\ &+ \left(\cosh(a\zeta) - \cos(b\zeta)\right) F + \left(\frac{\sinh(a\zeta)}{a} - \frac{\sin(b\zeta)}{b}\right) F^2 \\ &- \left. \frac{k_2}{|k_2|}\left(\frac{b - b\cosh(a\zeta)}{a} + \frac{a - a\cos(b\zeta)}{b}\right) G\right] V + P. \end{aligned} \tag{3.21}$$

In the case of $k_2 = 0$, a or b vanish which leads to the following approximate solutions as limits as a or b approach zero.

If a approaches zero, the limit is given by

$$x = \frac{m}{qb^2}\left[b^2\zeta I_4 + \left(1 - \cos(b\zeta)\right) F + \left(\zeta - \frac{\sin(b\zeta)}{b}\right) F^2\right] V + P, \tag{3.22}$$

and if b approaches zero, it is given by

$$x = \frac{m}{qa^2}\left[a^2\zeta I_4 - \left(1 - \cosh(a\zeta)\right) F - \left(\zeta - \frac{\sinh(a\zeta)}{a}\right) F^2\right] V + P. \tag{3.23}$$

If both approach zero, the limit is given by

$$x = \frac{m}{q}\left[\zeta I_4 - \frac{\zeta^2}{2}F - \frac{\zeta^3}{6}F^2\right] V + P. \tag{3.24}$$

To compare our solution (3.19) to the analytical solution given in (3.21) and its limits, we start with the same initial conditions and compute and plot both solutions in

Maple™[10] as if our solution were given by real numbers. Note that we use points $\tilde{p} = [p : 1] \in G_4$ in affine subspace G_4 in our approach and will only refer to p for better comparison This comparison is also a test whether special relativity is indeed encapsulated in our formalism.

The initial conditions need to be such that they could be translated into the finite field, so e. g. rational numbers with no multiples of a chosen prime number. The setting of the initial four-velocity can be a caveat since it has to fulfil the requirement $\eta(u) = -1$ where squares are involved which can lead to problems in the finite field. Properly treating this caveat, our formalism will give us a proper solution in the finite field. Yet, the comparison is much easier when using real numbers for both solutions.

We start by setting a constant electromagnetic field with electric field $\vec{E} = (E_x, E_y, E_z)^T$ and magnetic field $\vec{B} = (B_x, B_y, B_z)^T$. Furthermore, we set the initial velocity $u_0(x_0) = (u_0^0, \vec{u}^T)^T$ at the initial position $p = (0, 0, 0, 0)^T$ for simplicity. u_0^0 should be such that $\vec{u}^T\vec{u} = (u_0^0)^2 - 1$. We also initialise the mass m and the charge q of our body.

Afterwards, we let the computer calculate the trajectories for $\tau = 0$ to $\tau = \tau_{\max}$ according to (3.19) and (3.21) or equations (3.22), (3.23) and (3.24) if needed. Note that also in the analytical solution we compute only the values for $\tau \in \{0, \ldots \tau_{\max}\}$. We also make a spatio-temporal split of the solutions into $x = (t, \vec{x}^T)^T$. Additionally, we calculate deviations of our solution from the analytical one in the **relative spatial error**

$$\Delta x \coloneqq \frac{||\vec{x}_{\mathrm{an}} - \vec{x}_{\mathrm{FPP}}||}{||\vec{x}_{\mathrm{FPP}}||} \tag{3.25}$$

with $\vec{x}_{\mathrm{an}}$ and $\vec{x}_{\mathrm{FPP}}$ the spatial parts of the analytical solution and the solution in the framework of Finite Projective Physics, respectively, and the **relative temporal error**

$$\Delta t \coloneqq \frac{|t_{\mathrm{an}} - t_{\mathrm{FPP}}|}{t_{\mathrm{FPP}}} \tag{3.26}$$

with respect to proper time τ. These relative errors are chosen for a better comparability among different examples which may lead to bigger or smaller distances in spacetime and, thus, probably to bigger or smaller absolute errors, respectively.

We will present a few examples to show whether our approach is comparable to the standard ansatz. Note however that due to higher computational time we will usually work with $\tau_{\max} = 200$, i. e. 200 hoppings. For better visibility, we will only display every tenth data point of both trajectories and show the rest of them as a line. At first, we will look at the cases where either the electric or the magnetic field vanishes.

Hereinafter, we will use

$$u_0 = \left(\sqrt{2v_0^2 + 1}, v_0, 0, v_0\right)^T \tag{3.27}$$

as initial mean velocity where v_0 can be modified in order to achieve different results.

Vanishing Electric Field $\vec{E} = 0$ We start with a vanishing electric field $\vec{E} = 0$ and a magnetic field solely in z-direction, i. e.

$$\vec{B} = (0, 0, 10)^T .$$

The mass of the body is $m = 100$ with charge $q = 1$. Its initial velocity component v_0 is set to $v_0 = \frac{1}{200}$. This should result in a rather non-relativistic, spiralling motion around an axis parallel to the z-axis due to the magnetic field and the tilted and rather small initial velocity. Relativistic effects should be seen for high velocities and non-vanishing electric fields since these also change the magnitude of the velocity. Note that in this configuration we needed to apply (3.22) since the parameter a is vanishing.

The computed behaviour is shown in Figure 3.2a and is as expected. We also see that Figure 3.2c is a straight line and very close to $t(\tau) = \tau$. This is to be expected since there is no electric field which would couple to the zeroth component of the velocity due to $F_L = \tilde{F}u_0$ and, thus, would not lead to a change in the zeroth component of position. Additionally, the initial velocity is non-relativistic. This can also be seen by the relative temporal error being zero as shown in Figure 3.2d which means that our result exactly matches the analytical result. The relative spatial error as shown in Figure 3.2b is rather small and seems to be decreasing over multiple steps. Note however that the absolute spatial error is increasing over proper time τ. This will lead to a visible splitting of both trajectories for bigger τ.

In this non-relativistic regime without any electric field and a small magnetic field, we see that our approach works perfectly fine up to some relatively small deviations.

This changes if we increase the strength of the magnetic field to $B_z = 100$ and v_0 to $v_0 = \frac{1}{2}$ and additionally decrease the mass to $m = 10$. We expect a spiral-like structure in z-direction. Since the velocity is large and we only compute the trajectory for a finite amount of proper times, the structure will not appear as smooth as in Figure 3.2a. The result is shown in Figure 3.3. The temporal error is again exactly 0 but the temporal component of the trajectory as seen in Figure 3.3c has a higher slope than the one before. This is due to the relativistic velocity with $v_0 = \frac{1}{2}$.

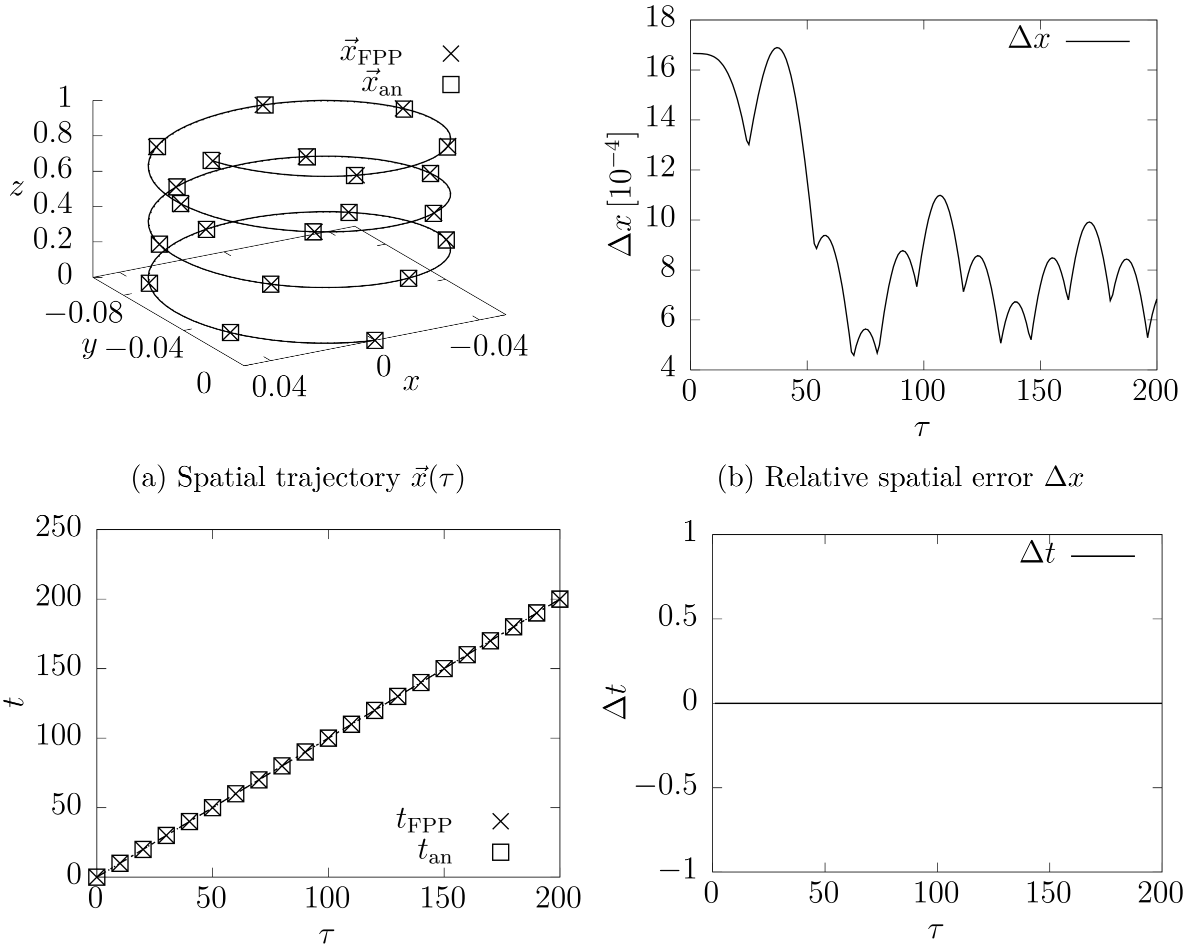
The spatial trajectory depicted in Figure 3.3a reveals some problems of this finite approach. Even though our solution gives a spiral in z-direction, as does the analytical solution, the diameter of this spiral differs rather drastically which is also reflected in the relative spatial error as seen in Figure 3.3b.

The reason behind this behaviour can be found in the hopping-ansatz of our approach. Since the distance of this hopping is only calculated from data known at finitely many points unlike in the case of differential equations where even an infinitesimal piece of proper time contributes to the shape of the trajectory. Here, the initial velocity is so high that it dominates the motion and can only be corrected at the next point where the same will happen analogously. Thus, in this case, our hopping ansatz overshoots the hopping distance which leads to a star-like structure bigger than the one of the analytical solution, but, in shape, a similar behaviour.

Vanishing Magnetic Field $\vec{B} = 0$ The next case we want to inspect is the case of a vanishing magnetic field, i. e. $\vec{B} = 0$. We set the electric field solely in x-direction to

$$\vec{E} = (5, 0, 0)^T$$

for simplicity. To get some relativistic effects, we increase the velocity to $v_0 = \frac{1}{20}$ and again use $m = 100$ and $q = 1$.



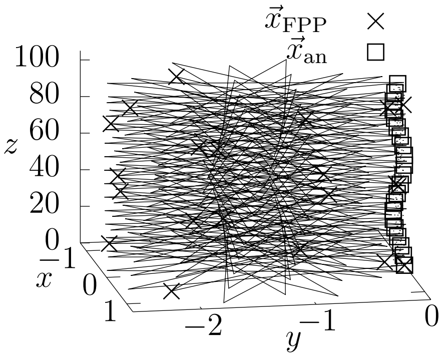
(a) Spatial trajectory $\vec{x}(\tau)$

(b) Relative spatial error Δx

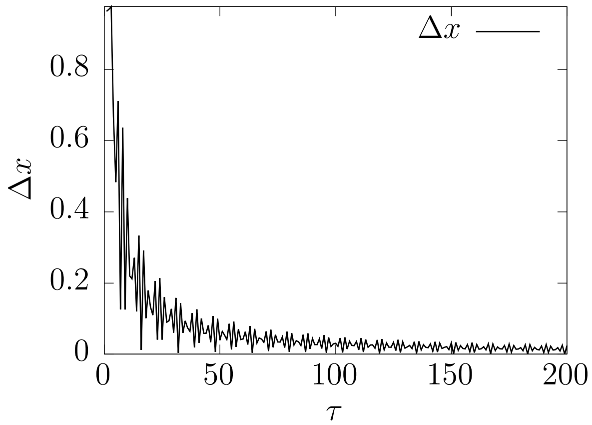
(c) Temporal trajectory $t(\tau)$

(d) Relative temporal error Δt

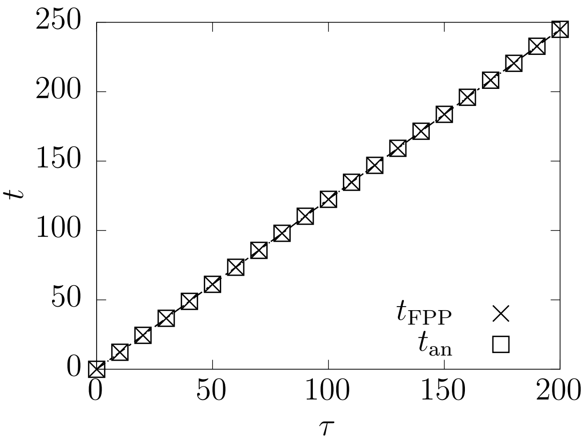
Figure 3.2: **Vanishing Electric Field $\vec{E} = 0$:** Spatial part $\vec{x}$ and temporal part t of the analytical solution x_{an} and the finite projective solution x_{FPP} and its relative spatial and temporal error Δx and Δt, respectively, of the trajectory of a body with mass $m = 100$, charge $q = 1$ and initial velocity $u_0 = \left(\sqrt{2v_0^2+1}, v_0, 0, v_0\right)^T$ with $v_0 = \frac{1}{200}$ at the origin, subject to an electric field $\vec{E} = (0,0,0)^T$ and a magnetic field $\vec{B} = (0,0,10)^T$.



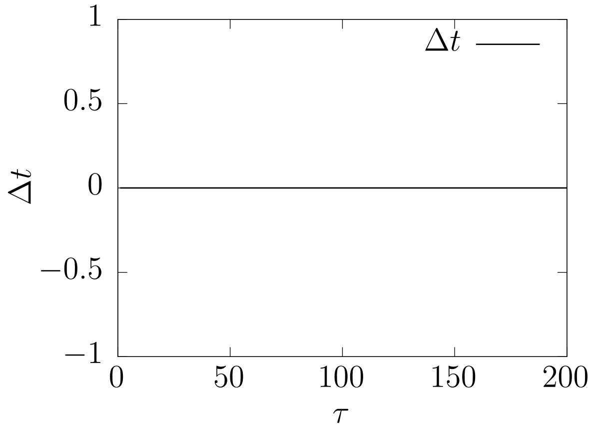
(a) Spatial trajectory $\vec{x}(\tau)$



(b) Relative spatial error Δx



(c) Temporal trajectory $t(\tau)$



(d) Relative temporal error Δt

Figure 3.3: **Vanishing Electric Field $\vec{E} = 0$:** Spatial part $\vec{x}$ and temporal part t of the analytical solution x_{an} and the finite projective solution x_{FPP} and its relative spatial and temporal error Δx and Δt, respectively, of the trajectory of a body with mass $m = 10$, charge $q = 1$ and initial velocity $u_0 = \left(\sqrt{2v_0^2+1}, v_0, 0, v_0\right)^T$ with $v_0 = \frac{1}{2}$ at the origin, subject to an electric field $\vec{E} = (0,0,0)^T$ and a magnetic field $\vec{B} = (0,0,100)^T$.

We expect an accelerated motion in x-direction due to the electric field in this direction, and a non-linear graph for the time component of the trajectory. The computed trajectories and their relative spatial and temporal errors are depicted in Figure 3.4. With these initial condition we needed to take (3.23) for the analytical solution.

We can immediately see that we are in a highly relativistic regime due to the extremely high time dilation. This is due to the accelerating effects of the electric field in contrast to the magnetic field which only changes the direction of the velocity. Even though the relative spatial and temporal error are rather small, they increase after $\tau \approx 60$ as shown in Figure 3.4b and Figure 3.4d, respectively.

This means that our approach produces relativistic results but seems to deviate further from the analytical solution in the highly relativistic regime over a high amount of time steps. This might also be due to the structure of the theory with only finite time steps $\Delta\tau$. Whereby in the infinitesimal approach of differential calculus even an infinitesimal time step $\mathrm{d}\tau$ leads to a change in the trajectory, here deviations cannot be corrected that easily by small time intervals since the next neighbour is sitting on a comparably big straight line as we calculated and not on a curved line with infinitesimal straight lines. This leads to a bigger error the bigger the step from one to another point is. Since the distances between two point grow rather large in this situation, this may explain the shown behaviour.

Non-vanishing Electric and Magnetic Field If we turn on the electric and the magnetic field to e. g.

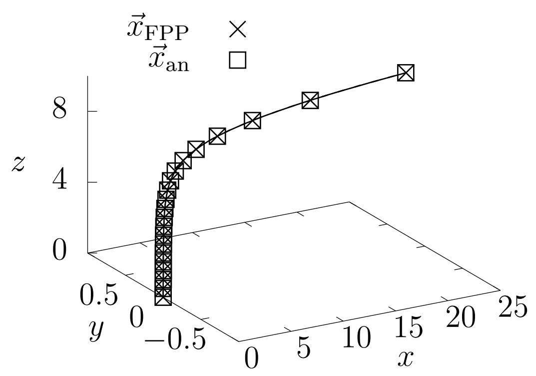
$$\vec{E} = (4, 1, 0)^T \text{ and } \vec{B} = (0, 0, 10)^T,$$

and use $q = 1$, $m = 100$ and $v_0 = \frac{1}{20}$, we should expect an accelerated motion with a spirally structure around some tilted axis partly in z-direction due to the magnetic field. In contrast to the first example in Figure 3.2 we expect a non-linear graph for the temporal trajectory with $t(\tau_{\max}) > \tau_{\max}$ due to time dilation as known from special relativity. Also, with these initial conditions we had to refer to (3.22) for the analytical solution.

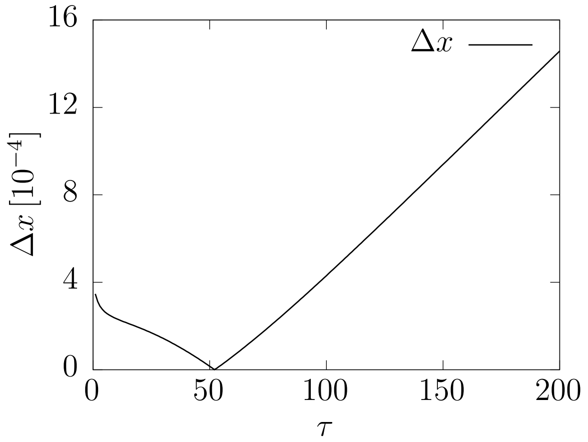
The results for this configuration are shown in Figure 3.5. The trajectories in Figure 3.5a and Figure 3.5c are as expected. The trajectories show rather small deviations which can also be seen by the small relative spatial and temporal errors as seen in Figure 3.5b and Figure 3.5d, respectively. It can be seen again that the errors become smaller the bigger τ gets on average and seem to stay on the same level the further we go. This may also be partly because of the large distance covered by the trajectory. Note that the wavy structure of the errors is due to the absolute values of the error since the error might be in principle positive or negative as our solution sometimes overshoots and sometimes undershoots.

It should be mentioned that the absolute error increases with increasing τ in the examples shown.

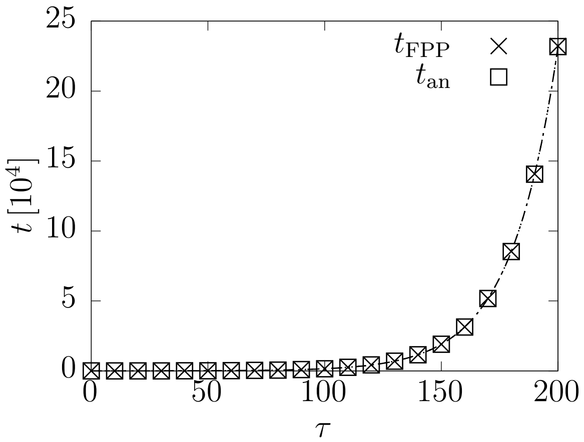
These errors are rather marginal but not vanishing. They are to be expected since our approach only yields a line per step $\Delta\tau$ in proper time in the trajectory and not a continuous trajectory as in the case of the analytical solution. This can be compared



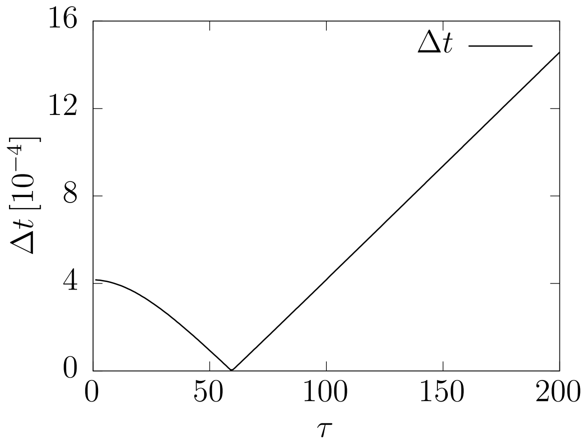
(a) Spatial trajectory $\vec{x}(\tau)$



(b) Relative spatial error Δx



(c) Temporal trajectory $t(\tau)$



(d) Relative temporal error Δt

Figure 3.4: **Vanishing Magnetic Field $\vec{B} = 0$:** Spatial part $\vec{x}$ and temporal part t of the analytical solution x_{an} and the finite projective solution x_{FPP} and its relative spatial and temporal error Δx and Δt, respectively, of the trajectory of a body with mass $m = 100$, charge $q = 1$ and initial velocity $u_0 = \left(\sqrt{2v_0^2 + 1}, v_0, 0, v_0\right)^T$ with $v_0 = \frac{1}{20}$ at the origin, subject to an electric field $\vec{E} = (5, 0, 0)^T$ and a magnetic field $\vec{B} = (0, 0, 0)^T$.

to the *Euler method* in numerical analysis for solving a differential equation since we essentially add the first derivative, i. e. the velocity, at each point. Note however the subtlety that our translation vectors u^+ are not quite the mean velocity u_0 which can be compared to the tangent vectors of the trajectory. Since we do not use infinitesimal segments of time as used in differentiation but only time steps of $\Delta\tau = 1$, we can never expect to get an error-less solution besides the one of a free particle.

If we change the mass to $m = 10$, this behaviour changes and our solution obviously differs from the analytical one as shown in Figure 3.6. It can be seen that the periodic behaviour due to the presence of the magnetic field has different periodicities. We can make out a point where these two different periods meet again. This can be seen in Figure 3.6b and Figure 3.6d since both show a dip in this region at around $\tau = 109$ and rise again afterwards. This periodical behaviour is expected to continue for larger τ. The relative errors decrease with increasing τ but the absolute errors form a rather periodic behaviour with non-decreasing amplitude. Note that the relative errors here are rather large and only get damped due to their relative nature.

Remark 3.2.8. Note that a change in the mass of the body is equivalent to the inverse change of the charge since only the fraction $\frac{q}{m}$ contributes in the calculations. Thus, if we divide the mass by 10, this is equivalent to multiplying the charge by 10 with an unchanged mass.

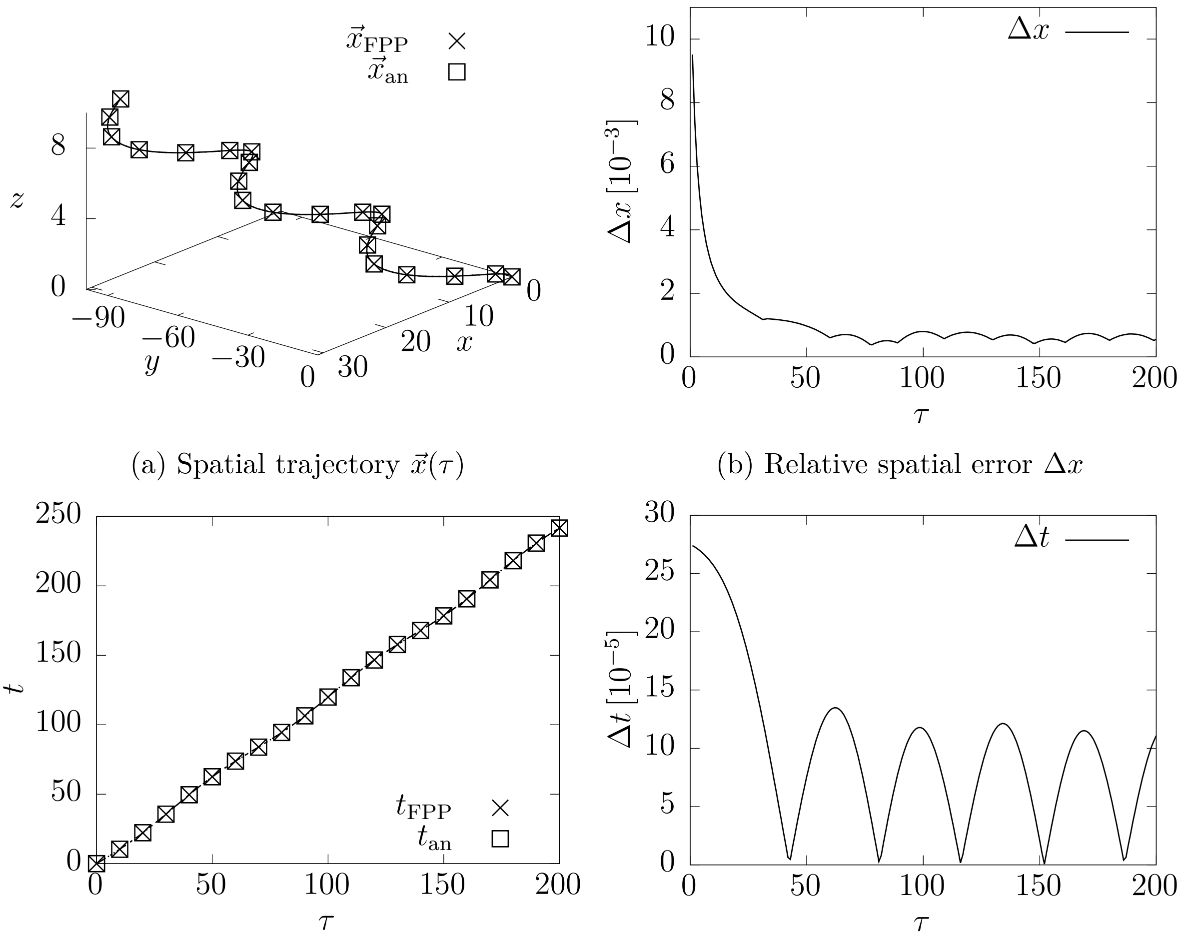
An explanation of this changed behaviour due to a change in the mass m would be that the smaller mass leads to a stronger change in the velocity by Newton's law $a = m^{-1}F$ if the force stays the same. This more rapid change of the velocity is not captured in our approach due to the finite nature of proper time.

Up to now, we have only used the two limit cases of (3.22) and (3.23). To show that the Finite Projective Physics framework also works in the general case up to some systematical errors or biases, we will use $\vec{E} = (4, 1, 0)^T$ as before but use $\vec{B} = (2, 0, 10)^T$ as the magnetic field. With these, we can use (3.21) for the calculation of the analytical solution of the trajectory. The mass and charge are set to $m = 10$ and $q = 1$, respectively. The structure of the velocity u_0 stays the same with $v_0 = \frac{1}{20}$.

The results are shown in Figure 3.7.

Optically, there is no difference between the two curves in Figure 3.7a and Figure 3.7c, respectively. We see that we run into a highly relativistic regime after $\tau \approx 60$ which also leads to a linear increase in the relative spatial and temporal error as can be seen Figure 3.7b and Figure 3.7d, respectively. Up to this point, the relative errors are actually decreasing, but, as seen earlier, highly relativistic motion leads to larger errors in our approach due to higher velocities.

Conclusion We have seen that the ansatz using the Finite Projective Physics framework yields results which agree highly on the structure of the trajectory with an analytical solution. Problems arise if we move to the highly relativistic regime, so for high electric fields or big initial velocities. Most of the problems can be explained by the finite nature of this approach, i. e. that the time step from one point to the next one



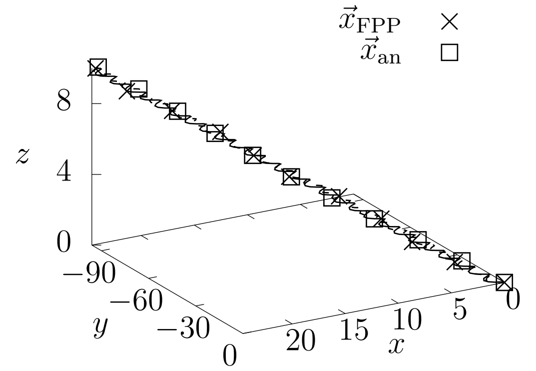
(a) Spatial trajectory $\vec{x}(\tau)$

(b) Relative spatial error Δx

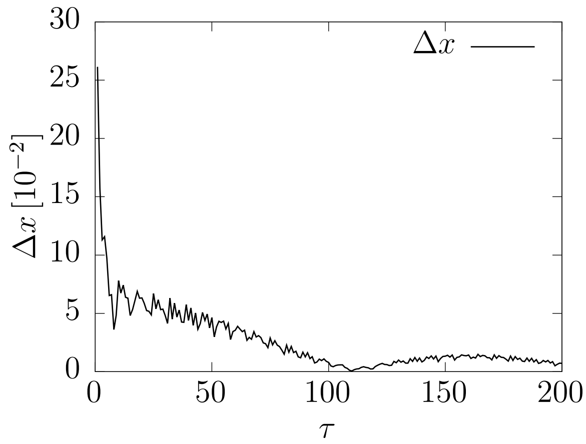
(c) Temporal trajectory $t(\tau)$

(d) Relative temporal error Δt

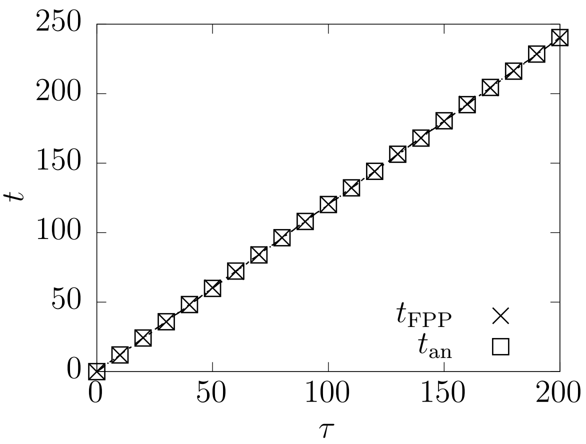
Figure 3.5: **Non-vanishing Electric and Magnetic Field:** Spatial part $\vec{x}$ and temporal part t of the analytical solution x_{an} and the finite projective solution x_{FPP} and its relative spatial and temporal error Δx and Δt, respectively, of the trajectory of a body with mass $m = 100$, charge $q = 1$ and initial velocity $u_0 = \left(\sqrt{2v_0^2 + 1}, v_0, 0, v_0\right)^T$ with $v_0 = \frac{1}{20}$ at the origin, subject to an electric field $\vec{E} = (4, 1, 0)^T$ and a magnetic field $\vec{B} = (0, 0, 10)^T$.



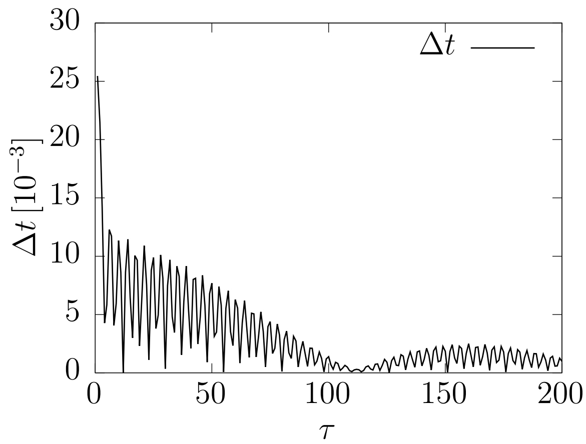
(a) Spatial trajectory $\vec{x}(\tau)$



(b) Relative spatial error Δx

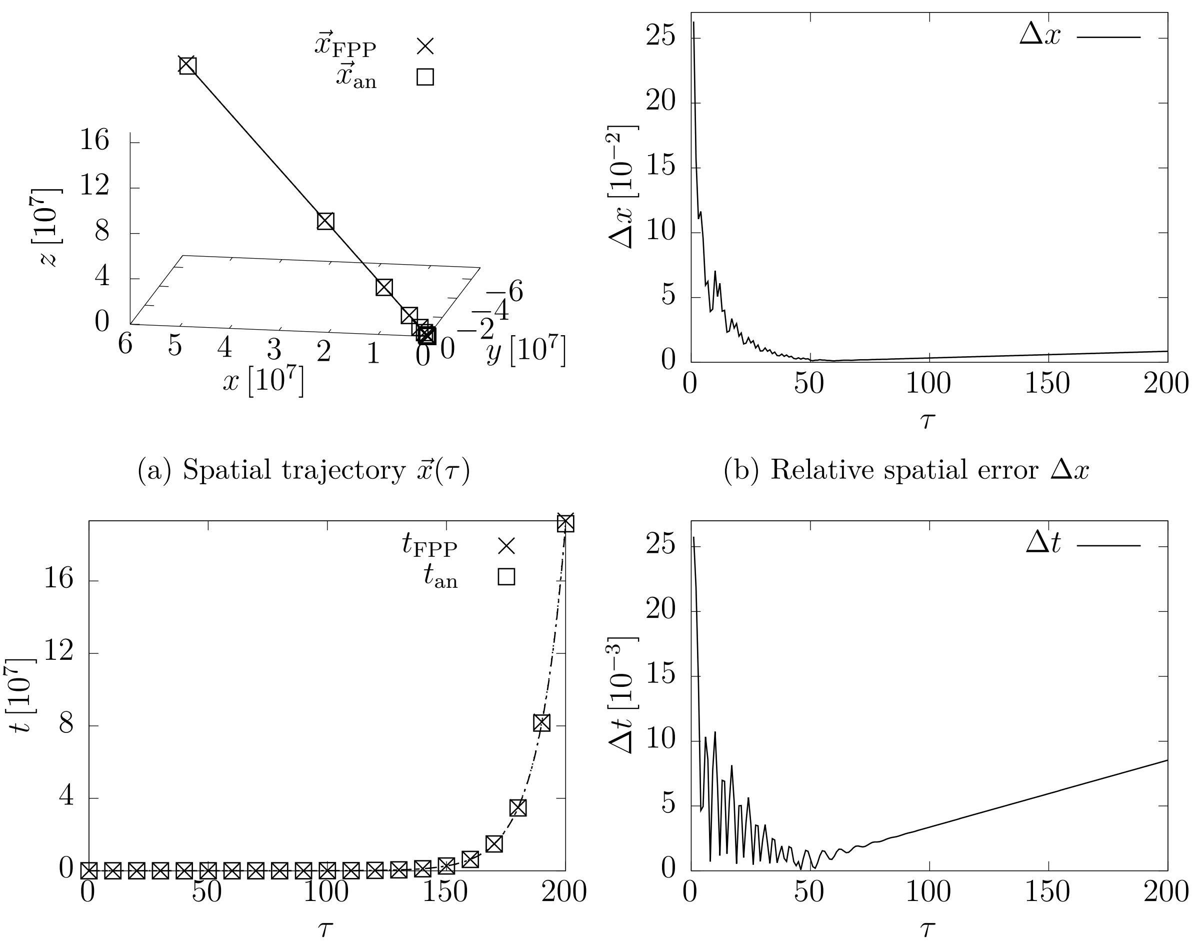


(c) Temporal trajectory $t(\tau)$



(d) Relative temporal error Δt

Figure 3.6: **Non-vanishing Electric and Magnetic Field:** Spatial part $\vec{x}$ and temporal part t of the analytical solution x_{an} and the finite projective solution x_{FPP} and its relative spatial and temporal error Δx and Δt, respectively, of the trajectory of a body with mass $m = 10$, charge $q = 1$ and initial velocity $u_0 = \left(\sqrt{2v_0^2 + 1}, v_0, 0, v_0\right)^T$ with $v_0 = \frac{1}{20}$ at the origin, subject to an electric field $\vec{E} = (4, 1, 0)^T$ and a magnetic field $\vec{B} = (0, 0, 10)^T$.



(a) Spatial trajectory $\vec{x}(\tau)$

(b) Relative spatial error Δx

(c) Temporal trajectory $t(\tau)$

(d) Relative temporal error Δt

Figure 3.7: **Non-vanishing Electric and Magnetic Field:** Spatial part $\vec{x}$ and temporal part t of the analytical solution x_{an} and the finite projective solution x_{FPP} and its relative spatial and temporal error Δx and Δt, respectively, of the trajectory of a body with mass $m = 10$, charge $q = 1$ and initial velocity $u_0 = \left(\sqrt{2v_0^2 + 1}, v_0, 0, v_0\right)^T$ with $v_0 = \frac{1}{20}$ at the origin, subject to an electric field $\vec{E} = (4, 1, 0)^T$ and a magnetic field $\vec{B} = (2, 0, 10)^T$.

of a trajectory is not infinitesimal which would lead to a continuous curve but rather finite with $\Delta\tau = 1$. This leads to phenomena known from numerical analysis, e. g. in the Euler method.

Nevertheless, our ansatz works rather well on small time scales if the finite time step does not lead to further problems as in the case of Figure 3.3. This is to be expected since it can only mimic the analytical solution if the time evolution within one proper time step is not too drastic. Thus, in the highly relativistic regime of high velocities, the change within one proper time step is overestimated and leads to higher deviations. These could be corrected by a smaller time scale with a re-definition of proper time such that in one step the change of motion is smaller. This may then lead to other errors, but this is beyond the scope of this thesis.

3.2.3 Non-Constant Electromagnetic Fields

In principle, our ansatz also works for non-constant electromagnetic fields as shown in (3.18), but there is up to now no set of limiting equations for the electromagnetic field like Maxwell's equations. The only condition is that the Lorentz force and its induced translations of the (bi-)quadric field yield another causal (bi-)quadric field.

Another condition is that the resulting mean velocity u_0 at each point of the trajectory fulfils the condition $u_0^T \eta u_0 = -1$. Since the mean velocities of neighbouring points on the trajectory are connected via the transformation $C\left((\tilde{F}(x_i)\right)$, these transformations should then be Lorentz transformation because these preserve the pseudo-norm of u_0. This requirement leads to an interesting condition on the electromagnetic fields.

Proposition 3.2.9. Let $\mathbb{F}_q$ be a finite field of odd order $q \in \mathbb{N}$ and let $(\mathbb{F}_q\mathrm{P}^4, Q)$ be a flat 4-dimensional spacetime which is shifted from a vacuum spacetime by translations corresponding to the Lorentz force field $F_L\colon \mathbb{F}_q\mathrm{P}^4 \to \mathbb{F}_q^4$ with respect to $H_\infty = \{x \in \mathbb{F}_q\mathrm{P}^4 \mid x_4 = 0\}$ which couples to a body with mass $m \in \mathbb{F}_q^\times$ and charge $\tilde{q} \in \mathbb{F}_q$ via the Faraday matrix field $\tilde{F}\colon \mathbb{F}_q\mathrm{P}^4 \to \mathrm{Mat}(4 \times 4, \mathbb{F}_q)$. Let $x\colon \mathbb{Z} \to \mathbb{F}_q\mathrm{P}^4$ be the trajectory of this body with initial mean velocity $u_0(x_0) \in \mathbb{F}_q^4$ at $x_0 \in G_4 = \mathbb{F}_q\mathrm{P}^4 \setminus H_\infty$.

The matrix $C(\tilde{F}(x_i)) = \left(I_4 - \frac{\tilde{q}}{2m}\tilde{F}(x_i)\right)^{-1}\left(I_4 + \frac{\tilde{q}}{2m}\tilde{F}(x_{i-1})\right)$ for $i \in \mathbb{Z}$ is a Lorentz transformation if

$$\tilde{F}(x_i)\eta\tilde{F}(x_i)^T = \tilde{F}(x_{i-1})\eta\tilde{F}(x_{i-1})^T. \tag{3.28}$$

Proof. We start by inspecting the condition $C(\tilde{F}(x_i))^T\eta C(\tilde{F}(x_i)) \stackrel{!}{=} \eta$.

$$\begin{aligned}\eta &\stackrel{!}{=} C(\tilde{F}(x_i))^T\eta C(\tilde{F}(x_i))\\ &= \left(I_4 + \frac{\tilde{q}}{2m}\tilde{F}(x_{i-1})^T\right)\left(I_4 - \frac{\tilde{q}}{2m}\tilde{F}(x_i)^T\right)^{-1}\eta\left(I_4 - \frac{\tilde{q}}{2m}\tilde{F}(x_i)\right)^{-1}\left(I_4 + \frac{\tilde{q}}{2m}\tilde{F}(x_{i-1})\right)\end{aligned}$$

Using that $\left(I_4 + \frac{\tilde{q}}{2m}\tilde{F}(x_{i-1})^T\right)$ and $\left(I_4 + \frac{\tilde{q}}{2m}\tilde{F}(x_{i-1})\right)$ are invertible since their eigenvalues are non-zero, this leads to

$$\begin{aligned}&\left(I_4 + \frac{\tilde{q}}{2m}\tilde{F}(x_{i-1})^T\right)^{-1} \eta \left(I_4 + \frac{\tilde{q}}{2m}\tilde{F}(x_{i-1})\right)^{-1} \\ &\stackrel{!}{=} \left(I_4 - \frac{\tilde{q}}{2m}\tilde{F}(x_i)^T\right)^{-1} \eta \left(I_4 - \frac{\tilde{q}}{2m}\tilde{F}(x_i)\right)^{-1}.\end{aligned}$$

Inverting both sides yields

$$\begin{aligned}&\left(I_4 + \frac{\tilde{q}}{2m}\tilde{F}(x_{i-1})\right) \eta \left(I_4 + \frac{\tilde{q}}{2m}\tilde{F}(x_{i-1})^T\right) \\ &\stackrel{!}{=} \left(I_4 - \frac{\tilde{q}}{2m}\tilde{F}(x_i)\right) \eta \left(I_4 - \frac{\tilde{q}}{2m}\tilde{F}(x_i)^T\right).\end{aligned}$$

Both sides are of similar structure and only involve the Faraday matrix at one point, respectively. We find that

$$\begin{aligned}&\left(I_4 \pm \frac{\tilde{q}}{2m}\tilde{F}(x_n)\right) \eta \left(I_4 \pm \frac{\tilde{q}}{2m}\tilde{F}(x_n)^T\right) \\ &= \eta \pm \frac{\tilde{q}}{2m}\left(\tilde{F}(x_n)\eta + \eta\tilde{F}(x_n)^T\right) + \frac{\tilde{q}^2}{4m^2}\tilde{F}(x_n)\eta\tilde{F}(x_n)^T\end{aligned}$$

for $n \in \{i-1, i\}$. A straightforward calculation shows that $\eta\tilde{F}(x_n)^T = -\tilde{F}(x_n)\eta$ for all $n \in \mathbb{Z}$. This is due to the structure of the Faraday matrix.

Therefore, the above condition reduces to

$$\eta + \frac{\tilde{q}^2}{4m^2}\tilde{F}(x_{i-1})\eta\tilde{F}(x_{i-1})^T \stackrel{!}{=} \eta + \frac{\tilde{q}^2}{4m^2}\tilde{F}(x_i)\eta\tilde{F}(x_i)^T.$$

For $\tilde{q} \neq 0$, this is equivalent to

$$\tilde{F}(x_i)\eta\tilde{F}(x_i)^T \stackrel{!}{=} \tilde{F}(x_{i-1})\eta\tilde{F}(x_{i-1})^T$$

as desired. □

This means that the object $\tilde{F}\eta\tilde{F}^T$ needs to be constant along the trajectory. A direct calculation shows that

$$\tilde{F}\eta\tilde{F}^T = \begin{pmatrix} \vec{E}^T\vec{E} & \vec{E}\times\vec{B} \\ \vec{E}\times\vec{B} & \vec{B}^T\vec{B}I_3 - \vec{E}\vec{E}^T - \vec{B}\vec{B}^T \end{pmatrix} = \begin{pmatrix} ||\vec{E}||^2 & \vec{S} \\ \vec{S} & ||\vec{B}||^2 I_3 - (E_iE_j + B_iB_j) \end{pmatrix}$$

with $i, j = 1, 2, 3$ and the definitions

$$\vec{B} = (B_x, B_y, B_z)^T, \vec{E} = (E_x, E_y, E_z)^T, ||\vec{A}||^2 = A_x^2 + A_y^2 + A_z^2$$

and

$$\vec{S} := \vec{E}\times\vec{B}$$

using the usual cross product. As indicated by the notation, $\vec{S}$ should be interpreted as the *Poynting vector* in natural units.

Inspecting the condition for $\tilde{F}\eta\tilde{F}^T$ needing to be constant along the trajectory block by block leads in particular to the conditions

$$||\vec{E}(x_i)||^2 = ||\vec{E}(x_{i-1})||^2 \text{ and } \vec{S}(x_i) = \vec{S}(x_{i-1}) \tag{3.29}$$

for all $i \in \mathbb{Z}$.

It seams unphysical to assume that, for a general system, the electric field has at each point of the trajectory the same magnitude and the Poynting vector is exactly the same at each point of the trajectory. Thus, we may conclude that our ansatz for electrodynamics is at least incomplete in the sense that the flat and vacuum background spacetime and a mere translation due to the presence of the Lorentz force is not enough to yield proper electromagnetism.

But, there may be a solution. Since the electromagnetic field and the moving body carry an energy-momentum-stress-tensor, they change the spacetime by Einstein's equation of General Relativity. Hence, our assumption that the (bi-)quadric field is only given by translations with respect to the affine subspace G_4 may be not general enough. We would need to incorporate this change curvature given by Einstein's equations into our description of the spacetime probably by an additional tilt of the hyperplane at infinity at each point. This would need a description of General Relativity in the framework of Finite Projective Physics which has not been done yet and is beyond the scope of this thesis.

3.3 Gauge Transformations

We have already mentioned that electrodynamics or quantum electrodynamics is described in the framework of *gauge theories*. In modern physics, electrodynamics and many other physical field theories or quantum field theories as used in the Standard Model of Particle Physics are formulated in such a way. This means that the theory, or more precisely the Lagrangian of the theory, is invariant under some transformation group, the *gauge group* G, acting locally on the fields. This invariance is usually achieved by the introduction of a *gauge field* $A^\mu \in \mathrm{Lie}(G)$ which transforms in the right way to cancel the transformational behaviour of the field in the Lagrangian. This gauge field or its quantization will lead to particles which carry and transfer the induced interaction described by the theory. Note that the gauge group directly influences the type of interaction the theory will describe. Technically, this is achieved by means of principal bundles and the corresponding connection on this bundle.

In the case of electrodynamics, the gauge group is given by the one-dimensional unitary group $G = U(1)$ and the gauge field A which will later lead, after some technicalities, to the photon which transfers the electromagnetic interaction. In the Standard Model of Particle Physics, the gauge group is given by $SU(3) \times SU(2) \times U(1)$ and the gauge fields will later become gluons, $Z^\pm$-, W^0-bosons and the photon. This theory describes the strong and the electroweak interaction which is the unification of the weak and the electromagnetic interaction, and is very well tested.

Our aim is to find similar structures in the framework of Finite Projective Physics, in particular gauge groups that arise naturally. These could then be used to form suitable physical theories as analogues to the standard gauge theories. Up to now, there is no algorithm to turn a gauge group into a physical theory which describes the desired interaction in this framework.

We will inspect different geometrical objects and their symmetry groups, in particular the intersection of two quadrics. In the case of translated quadrics, this intersection can be decomposed into elements of a *hyperplane of intersection* and of the hyperplane at infinity. Therefore, we will also consider symmetries or transformations of these hyperplanes. Additionally, we will introduce the concept of *complete quadrangles* which could be used to incorporate a geometrical form of a gauge theory in our approach. In the end, we will consider an idea of how to introduce charges corresponding to some form of interaction into this theory by means of invariant matrices.

3.3.1 Intersections of Quadrics and their Symmetries

Our first attempt at suitable gauge group structures are the symmetry groups of intersections of two quadrics of a quadric field. Since the interaction in a gauge theory is essentially described on the intersection of two chart patches of the principal bundle where coordinate systems on the fibres may differ and will have to be related to each other by the proper connection, the right analogue in the Finite Projective Physics framework might be the intersections of quadrics and their symmetry groups since all dynamical information is stored in these quadrics. Furthermore, these intersections will lead to points which are neighbours of both quadric locations, i. e. intermediate steps from one to the other. Here, the interaction could take place and change the trajectory accordingly. Note that these intersections might lead to points in the chosen affine subspace, i. e. the physical space, and in the hyperplane at infinity. Latter might be regarded as massless particle states if the hyperplane is not tilted, as our theory up to now only considered massive bodies in the affine subspace which could be incorporated by setting the last coordinate in the affine space to the mass instead of 1. If the hyperplane is tilted, this could be regarded as a mixing of massless and massive particle states.

Let us now consider two non-degenerate quadrics $Q, R \subseteq \mathbb{F}_q\mathrm{P}^n$ with symmetric representation matrices $M, N \in \mathrm{Mat}\,((n+1) \times (n+1), \mathbb{F}_q)$, respectively. We will generalize from [15] from quadrics which are only changed by a translation with respect to a chosen hyperplane at infinity to more general forms. For simplicity, we will mostly choose $R = H^{\pm}$ such that $N = \eta^{\pm}$, i. e. one of the the standard Minkowski forms. Remember that this transformation is locally always possible in the finite projective quadric field setting.

The intersection of the quadrics Q and R is given by the algebraic (projective) variety $Q \cap R = \{x \in \mathbb{F}_q\mathrm{P}^n \mid x^T M x = x^T N x = 0\}$, i. e. the zero locus of two homogeneous polynomials of degree two. An illustration of the situation is shown in Figure 3.8. The computation of the intersection can be reduced to the computation of the intersection of one of the non-degenerate quadrics with a degenerate one. This is done as follows.

At first, we build the pencil of quadrics $\{Q_\lambda \mid \lambda \in \mathbb{F}_q\}$ which corresponds to the set

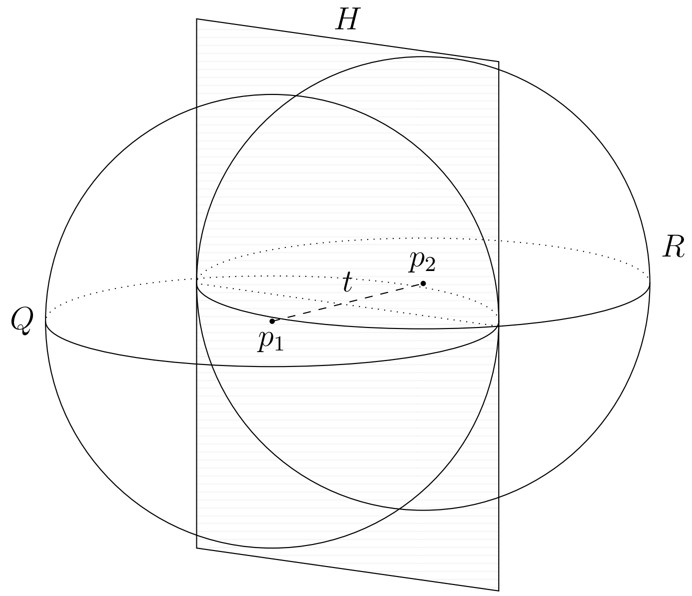


Figure 3.8: Illustration of two quadrics Q, R with centres p_1, p_2, respectively, as three-dimensional spheres in the affine subspace which are translated by the translation vector t with respect to each other. The plane of intersection in the affine subspace is illustrated by H.

of matrices $\{A_\lambda = M + \lambda N \mid \lambda \in \mathbb{F}_q\}$. Points of the intersection $Q \cap R$ are necessarily elements of every Q_λ in this pencil since $x^T (M + \lambda N)\, x = x^T M x + \lambda x^T N x = 0 + 0 = 0$ for points in the intersection. Thus, for easier calculations it suffices to search for degenerate quadrics in this pencil, i. e. $\lambda \in \mathbb{F}_q$ such that $\det (M + \lambda N) = 0$. Since N is by assumption invertible, this reduces to an eigenvalue problem:

$$\det (M + \lambda N) = 0 \iff \det \left(-MN^{-1} - \lambda I_n\right) = 0.$$

Remark 3.3.1. Note that in full projective fashion this would have to be formulated as $\mu M + \lambda N$ for $\mu, \lambda \in \mathbb{F}_q$ not both zero. But, since we are interested in degenerate quadrics in this pencil, the case of one of μ, λ being equal to zero can be excluded because of the assumption of Q, R being non-degenerate.

After we have found these eigenvalues of $E := -MN^{-1}$, we can choose one of them and compute the elements of this quadric. Since it is degenerate, the condition of a point being in this quadric is usually simpler than the one of being a point in Q or R, e. g. the points have to lie in certain hyperplanes. The points of the intersection $Q \cap R$ are then the points of the intersection $Q \cap Q_\lambda$ or $R \cap Q_\lambda$.

In some cases, a decomposition of the matrix A_λ corresponding to the chosen eigenvalue can be achieved to find the points of the intersection in certain hyperplanes, i. e.

$$A_\lambda = l h^T + h l^T \tag{3.30}$$

with $l, h \in \mathbb{F}_q^{n+1}$ the normal vectors of these hyperplanes. In this case, the intersection is given by points $x \in \mathbb{F}_q\mathrm{P}^n$ of the quadric Q or R which also lie in one of these hyperplanes,

i. e.

$$x \in Q \wedge \left(h^T x = 0 \vee l^T x = 0\right). \tag{3.31}$$

Example 3.3.2. Let us consider the case $R = H^+$ corresponding to $N = \eta^+$, and Q corresponding to

$$M = \left(T^{-1}\right)^T \eta^+ T^{-1}, T = \begin{pmatrix} I_n & t \\ 0 & 1 \end{pmatrix} \in \mathrm{PGL}(n+1, \mathbb{F}_q) \tag{3.32}$$

with T a translation with respect to the affine subspace G_n and with translation vector $t \in \mathbb{F}_q^n \setminus \{0\}$.

Here, we have to find the eigenvalues of

$$E = -\left(T^{-1}\right)^T \eta^+ T^{-1} \left(\eta^+\right)^{-1} = -\left(T^{-1}\right)^T \eta^+ T^{-1} \eta^+ = \begin{pmatrix} -I_n & \eta t \\ t^T & -t^T \eta t - 1 \end{pmatrix}.$$

For $v \in \mathbb{F}_q^n$ with $t^T v = 0$, we find that

$$E \begin{pmatrix} v \\ 0 \end{pmatrix} = - \begin{pmatrix} v \\ 0 \end{pmatrix}$$

such that $\lambda = -1$ is an eigenvalue of E. Using this eigenvalue, we can compute

$$A_{\lambda=-1} = M - N = \begin{pmatrix} \eta & -\eta t \\ -t^T \eta & 1 + t^T \eta t \end{pmatrix} - \begin{pmatrix} \eta & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -\eta t \\ -t^T \eta & t^T \eta t \end{pmatrix}.$$

We can immediately see that points $x \in H_\infty = \mathbb{F}_q \mathrm{P}^n \setminus G_n$ are trivially points of the degenerate quadric Q_{-1} since they fulfil $x^T A_{-1} x = 0$ because of $x_n = 0$. Hence, we find that the intersection $Q \cap R$ contains all points of Q or R which lie in the hyperplane at infinity.

For points $p = [x : 1] \in G_n$ we find that

$$p^T A_{-1} p = \begin{pmatrix} x^T & 1 \end{pmatrix} \begin{pmatrix} 0 & -\eta t \\ -t^T \eta & t^T \eta t \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = t^T \eta t - 2 x^T \eta t \overset{!}{=} 0$$

which leads to the condition $h^T p = 0$ with

$$h^T = \begin{pmatrix} -(\eta t)^T & \frac{1}{2} t^T \eta t \end{pmatrix}. \tag{3.33}$$

We will call the hyperplane H associated to h the **hyperplane of intersection** in this case since it consists of the points of the intersection in the affine subspace.

Furthermore, we find that

$$A_{-1} = h_\infty h^T + h h_\infty^T \tag{3.34}$$

with $h_\infty^T = (0, \ldots, 0, 1) \in \mathbb{F}_q^{n+1}$ as a generalization of the results in [15]. This means that points in $Q \cap R$ are given by the points of Q or R which lie in the hyperplane H corresponding to h or in the hyperplane at infinity H_∞ corresponding to h_∞, i. e.

$$Q \cap R = (Q \cap H) \cup (Q \cap H_\infty). \tag{3.35}$$

Later, we will apply this technique to the two-dimensional case with a time-like or space-like translation corresponding to η^+ and η^-, respectively.

To find the symmetry group of the intersection of two quadrics, we can employ our $\mathrm{PGL}(n+1, \mathbb{F}_q)$-action again and extend it to varieties which are the common zero locus of two or even more homogeneous polynomials of degree two $p_1, \ldots, p_k \in \mathbb{F}_q[X_0, \ldots, X_n]^{\mathrm{hom},2}$, $k \in \mathbb{N}$, i. e.

$$S \triangleright \{x \in \mathbb{F}_q\mathrm{P}^n \mid p_1(x) = \ldots = p_k(x) = 0\} = \{x \in \mathbb{F}_q\mathrm{P}^n \mid p_1(S^{-1}x) = \ldots = p_k(S^{-1}x) = 0\}$$

for $S \in \mathrm{PGL}(n+1, \mathbb{F}_q)$. This will again lead to a variety by the same reasons as for quadrics. Note that this action can also be understood as a transformation on the underlying projective space, i. e. $\{x \in \mathbb{F}_q\mathrm{P}^n \mid p_1(S^{-1}x) = \ldots = p_k(S^{-1}x) = 0\} = \{Sy \in \mathbb{F}_q\mathrm{P}^n \mid p_1(y) = \ldots = p_k(y) = 0\}$.

Lemma 3.3.3. Let $\mathbb{F}_q$ be a finite field of odd order $q \in \mathbb{N}$ and let $Q, R \subseteq \mathbb{F}_q\mathrm{P}^n$ be non-degenerate quadrics with symmetric representation matrices $M, N \in \mathrm{Mat}((n+1) \times (n+1), \mathbb{F}_q)$.

The automorphism group $\mathrm{Aut}(Q \cap R) = \{f \in \mathrm{Aut}(\mathbb{F}_q\mathrm{P}^n) \mid f(Q \cap R) = Q \cap R\} = \{S \in \mathrm{PGL}(n+1, \mathbb{F}_q) \mid \forall x \in Q \cap R\colon Sx \in Q \cap R\} \subseteq \mathrm{PGL}(n+1, \mathbb{F}_q)$ of the intersection $Q \cap R$ is equivalent to the stabilizer $\mathrm{PGL}(n+1, \mathbb{F}_q)_{Q\cap R}$ of $Q \cap R$ under the $\mathrm{PGL}(n+1, \mathbb{F}_q)$-action $\triangleright$, i. e.

$$\mathrm{Aut}(Q \cap R) = \mathrm{PGL}(n+1, \mathbb{F}_q)_{Q\cap R}.$$

In particular, $\mathrm{PGL}(n+1, \mathbb{F}_q)_Q \cap \mathrm{PGL}(n+1, \mathbb{F}_q)_R \subseteq \mathrm{PGL}(n+1, \mathbb{F}_q)_{Q\cap R}$.

Proof. The automorphism group of the intersection $Q \cap R$ is defined as $\mathrm{Aut}(Q \cap R) = \{S \in \mathrm{PGL}(n+1, \mathbb{F}_q) \mid \forall x \in Q \cap R\colon Sx \in Q \cap R\} \subseteq \mathrm{PGL}(n+1, \mathbb{F}_q)$. This means that $S \in \mathrm{Aut}(Q \cap R)$ if for $x \in \mathbb{F}_q\mathrm{P}^n$ with $p_1(x) = x^T M x = 0$ and $p_2(x) = x^T N x = 0$ we also have $(Sx)^T M S x = (S^{-1} \triangleright p_1)(x) = (Sx)^T N S x = (S^{-1} \triangleright p_2)(x) = 0$, i. e. for $x \in Q \cap R$ we have $x \in S^{-1} \triangleright (Q \cap R)$. This means that $S \in \mathrm{Aut}(Q \cap R)$ is equivalent to $S^{-1} \in \mathrm{PGL}(n+1, \mathbb{F}_q)_{Q\cap R}$, i. e. S^{-1} is in the stabilizer of $Q \cap R$. With that, we found $\mathrm{Aut}(Q \cap R) = \mathrm{PGL}(n+1, \mathbb{F}_q)_{Q\cap R}$.

In particular, for $S \in \mathrm{PGL}(n+1, \mathbb{F}_q)_Q \cap \mathrm{PGL}(n+1, \mathbb{F}_q)_R$ we immediately see that $S \triangleright (Q \cap R) = \{x \in \mathbb{F}_q\mathrm{P}^n \mid (S \triangleright p_1)(x) = (S \triangleright p_2)(x) = 0\} = \{x \in \mathbb{F}_q\mathrm{P}^n \mid p_1(x) = (S \triangleright p_2)(x) = 0\} = \{x \in \mathbb{F}_q\mathrm{P}^n \mid p_1(x) = p_2(x) = 0\} = Q \cap R$. Hence, $S \in \mathrm{PGL}(n+1, \mathbb{F}_q)_{Q\cap R}$ and $\mathrm{PGL}(n+1, \mathbb{F}_q)_Q \cap \mathrm{PGL}(n+1, \mathbb{F}_q)_R \subseteq \mathrm{PGL}(n+1, \mathbb{F}_q)_{Q\cap R}$. Since both stabilizers of Q and R are subgroups of $\mathrm{PGL}(n+1, \mathbb{F}_q)$, respectively, their intersection is by results of group theory again a subgroup. □

Remark 3.3.4. Consider now this intersection $\mathrm{PGL}(n+1, \mathbb{F}_q)_Q \cap \mathrm{PGL}(n+1, \mathbb{F}_q)_R$. By the same reasoning as before $\mathrm{PGL}(n+1, \mathbb{F}_q)_Q = \mathrm{Aut}(Q)$ and analogously for R. For the case of $R = H^+$ and $Q = S \triangleright H^+$ for $S \in \mathrm{PGL}(n+1, \mathbb{F}_q)$, we have that $L \in \mathrm{PGL}(n+1, \mathbb{F}_q)_R$ if $L^T \eta^+ L = \eta^+$. We will call $\mathrm{Aut}(H^+)$ the **extended Lorentz group** and an element L of it a (generalized) **Lorentz transformation**. Similarly, we find that $M \in \mathrm{PGL}(n+1, \mathbb{F}_q)_Q$ if $(S^{-1}M)^T \eta^+ (S^{-1}M) = (S^{-1})^T \eta^+ S^{-1}$. This means that $M \in \mathrm{PGL}(n+1, \mathbb{F}_q)_Q \iff S^{-1}MS \in \mathrm{Aut}(H^+)$.

Now, a projectivity L lies in the intersection $\mathrm{PGL}(n+1,\mathbb{F}_q)_Q \cap \mathrm{PGL}(n+1,\mathbb{F}_q)_R$ if L and $S^{-1}LS$ are generalized Lorentz transformation, i. e. $L \in \mathrm{Aut}(H^+)$ and $S^{-1}LS = L' \in \mathrm{Aut}(H^+)$. This means that the intersection $\mathrm{PGL}(n+1,\mathbb{F}_q)_Q \cap \mathrm{PGL}(n+1,\mathbb{F}_q)_R$ is given by a subgroup $\Lambda \subseteq \mathrm{Aut}(H^+)$ which is stable under conjugacy with S^{-1}, i. e. $S^{-1}\Lambda S = \Lambda$.

Since quadrics over finite fields in finite dimension are finite, the intersections are as well. Therefore, the automorphism group of this intersection can be viewed as permutations on the points of the intersection. Thus, we may ask whether the symmetry group of this intersection allows every possible permutation of points in the intersection, i. e. whether we can find an isomorphism between $S_{|Q\cap R|}$, the symmetric group on $|Q \cap R|$ symbols, and $\mathrm{Aut}(Q \cap R) \subseteq \mathrm{PGL}(n+1,\mathbb{F}_q)$.

Hereinafter, we want to answer this question in the case of $n = 2$ and $R = H^\pm$ and $Q = T \triangleright H^\pm$ for a translation T with respect to the affine subspace G_n. For this, we will use the findings of 3.3.2.

Theorem 3.3.5. Let $\mathbb{F}_q$ be a finite field of odd order $q \in \mathbb{N}$ with -1 not a square in $\mathbb{F}_q$, and let $Q, R \subseteq \mathbb{F}_q\mathrm{P}^2$ be non-degenerate quadrics with $R = H^+$ corresponding to the symmetric representation matrix $\eta^+ \in \mathrm{Mat}(3\times 3,\mathbb{F}_q)$ and $Q = T \triangleright H^+$ with $T \in \mathrm{PGL}(3,\mathbb{F}_q)$ representing a translation with respect to $H_\infty = \{x \in \mathbb{F}_q\mathrm{P}^2 \mid x_2 = 0\}$ with translation vector $(s,t)^T \in \mathbb{F}_q^2$ with $-s^2 + t^2 \neq 0$.

If $(s^2-t^2)^2 - 4(s^2-t^2) \in \mathbb{F}_q^\times$ is a square in $\mathbb{F}_q$, then $Q \cap R = \{w_{1/2}, u_{1/2}\}$ with $w_{1/2} \in G_2 = \mathbb{F}_q\mathrm{P}^2 \setminus H_\infty$ and $u_{1/2} \in H_\infty$, and

$$\mathrm{Aut}(Q \cap R) \cong S_4.$$

Proof. We will use the results from 3.3.2 to find that in our case the intersection $Q \cap R$ is given by points of H^+ which lie in the hyperplane at infinity H_∞ or in the hyperplane H with normal vector $h = (s, -t, \frac{t^2-s^2}{2})^T$.

The points of $R \cap H_\infty$ are quickly identified. They are of the form $[x:y:0] \in H_\infty$ such that $-x^2+y^2+0^2 = 0$. Note that $y = 0$ or $x = 0$ would lead to $x = y = 0$ which is not included in the projective space. This means that $x, y \neq 0$ and $u_{1/2} = [\pm 1 : 1 : 0] \in R \cap H_\infty$.

To find the points of $R \cap H$, we inspect the condition $h^T p = 0$ for points $p = [x:y:0]$. In this case, we find that $sx = ty$ which leads to $-x^2+y^2 = \left(-\frac{t^2}{s^2}+1\right)y^2 = \frac{s^2-t^2}{s^2}y^2 = 0$ if $s \neq 0$. Since y cannot be zero since this would lead to $x = 0$, we find that $s^2 - t^2 = 0$ which we excluded by assumption. The case of $t \neq 0$ is analogous. If $s = 0$, the condition yields $y = 0$ which is excluded since this leads to $x = 0$.

Thus, we find that the points of $R \cap H$ do not lie at infinity, i. e. they are of the form $p = [x:y:1]$, and we find that $Q \cap R = \{u_{1/2}\} \cup (R \cap H)$. The condition $h^t p = 0$ then becomes $sx - ty = \frac{s^2-t^2}{2}$.

If $s = 0$, this yields $y = \frac{t}{2}$ and with that $x^2 = y^2 + 1 = \frac{t^2}{4} + 1$. Therefore, $x = \pm\frac{\sqrt{t^2+4}}{2}$. This root exists since by assumption $(s^2-t^2)^2 - 4(s^2-t^2) = t^2(t^2+4)$ is a square in $\mathbb{F}_q$. Since t^2 is a square and the squares form a subgroup of the group of units, we conclude that $t^2 + 4$ is a square in $\mathbb{F}_q$. Hence, we find

$$w_{1/2} = [\pm\frac{\sqrt{t^2+4}}{2} : \frac{t}{2} : 1] \in R \cap H.$$

If $s \neq 0$, we can re-arrange the condition to find $x = \frac{2ty+s^2-t^2}{2s}$. Inserting this into $-x^2 + y^2 + 1 = 0$ and simplifying yields

$$y_{1/2} = \frac{t(s^2-t^2) \pm \sqrt{(s^2-t^2)(s^2-t^2-4)}s}{2(s^2-t^2)}$$

corresponding to

$$x_{1/2} = \frac{s(s^2-t^2) \pm \sqrt{(s^2-t^2)(s^2-t^2-4)}t}{2(s^2-t^2)}$$

such that

$$w_{1/2} = [x_{1/2} : y_{1/2} : 1].$$

The square root exists and is non-zero by assumption. Otherwise, there would only be one point in this hyperplane or none in addition to the points $u_{1/2} \in H_\infty$ in the intersection $Q \cap R$.

We can now use these four points to find the automorphisms of $Q \cap R$ by identifying them with permutations of these four points. For simplicity, we define $w_1 := 1, w_2 := 2, u_1 := 3$ and $u_2 := 4$ such that we can naturally identify automorphisms with permutations of the symbols $\{1, 2, 3, 4\}$, i. e. elements of S_4. We will use the cycle notation used in the study of the symmetric group. For example, $(134) \in S_4$ corresponds to the permutation $2 \mapsto 2, 1 \mapsto 3 \mapsto 4 \mapsto 1$.

To find a suitable isomorphism between $\mathrm{Aut}(Q \cap R)$ and S_4, we will use a structural statement from group theory which states that the group S_n is generated by the cycles (12) and $(123 \dots n)$. Therefore, we need to find a projectivity which interchanges w_1 with w_2 and leaves u_1, u_2 invariant, and a projectivity with the mapping $w_1 \mapsto w_2 \mapsto u_1 \mapsto u_2 \mapsto w_1$.

If we can prove that the four points w_1, w_2, u_1, u_2 form a projective frame in any order, we can use the fact that given two projective frames there is exactly one projective linear transformations that maps one to the other, i. e. $(w_1, w_2, u_1, u_2) \mapsto (w_2, w_1, u_1, u_2)$ and $(w_1, w_2, u_1, u_2) \mapsto (w_2, u_1, u_2, w_1)$. This would then immediately lead to the desired isomorphism $\mathrm{Aut}(Q \cap R) \cong S_4$.

Since we have $4 = 2 + 2$ points, we need to show that 3 of them are the image of a basis under the canonical projection and the last one is the image of the sum of these basis vectors. At first, we need to find the right scaling of the points such that, e. g. $\tilde{w}_2 = \tilde{w}_1 + \tilde{u}_1 + \tilde{u}_2$ with $\tilde{x} \in \mathbb{F}_q^3$ a pre-image of $x \in \mathbb{F}_q\mathrm{P}^2$ under the canonical projection. Because the last entry of u_1 and u_2 is zero, both w_1 and w_2 need to have the same entry in the last coordinate. We choose for simplicity the scaling as above with a 1 in the last spot, i. e. $\tilde{w}_1 = (x_1, y_1, 1) \in \mathbb{F}_q^3$ and $\tilde{w}_2 = (x_2, y_2, 1)^T \in \mathbb{F}_q^3$. This means that we need to find $\mu, \lambda \in \mathbb{F}_q$ such that $x_2 = x_1 + \lambda - \mu$ and $y_2 = y_1 + \lambda + \mu$.

A solution is given by

$$\lambda = \frac{1}{2}(x_2 - x_1 + y_2 - y_1) = \frac{\sqrt{(s^2 - t^2)(s^2 - t^2 - 4)}}{2(t - s)}$$

and

$$\mu = \frac{1}{2}(x_1 - x_2 + y_2 - y_1) = -\frac{\sqrt{(s^2 - t^2)(s^2 - t^2 - 4)}}{2(t + s)}.$$

With these, we can now check whether w_1, u_1, u_2 are the pre-image of a basis of $\mathbb{F}_q^3$. We form the matrix

$$B = (\tilde{w}_1, \tilde{u}_1, \tilde{u}_2) = \begin{pmatrix} x_1 & \lambda & -\mu \\ y_1 & \lambda & \mu \\ 1 & 0 & 0 \end{pmatrix}$$

whose determinant determines whether $\{\tilde{w}_1, \tilde{u}_1, \tilde{u}_2\}$ forms a basis of $\mathbb{F}_q^3$ by not being equal to zero. We find

$$\det(B) = 2\lambda\mu = \frac{s^2 - t^2 - 4}{2} \neq 0 \text{ for } s^2 - t^2 \neq 4$$

which is true by assumption. Thus, $\{\tilde{w}_1, \tilde{u}_1, \tilde{u}_2\}$ forms a basis of $\mathbb{F}_q^3$ with $\tilde{w}_2 = \tilde{w}_1 + \tilde{u}_1 + \tilde{u}_2$. Hence, $\{w_1, w_2, u_1, u_2\}$ forms a projective frame.

Explicitly, the matrix representation of the projectivity corresponding to the permutation $(12) \in S_4$ is given by

$$M_{(12)} = \begin{pmatrix} -1 & 0 & s \\ 0 & -1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

and the one corresponding to the second generator $(1234) \in S_4$ can be calculated for example by

$$M_{(1234)} = CMC^{-1} \text{ with } C = (\tilde{w}_1, \tilde{w}_2, \tilde{u}_1) \text{ and } M = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Note that these matrices as well as their images are to be understood in the projective way. The whole group is then given by arbitrary products of $M_{(12)}$ and $M_{(1234)}$. □

Remark 3.3.6.

1. As a special case of the theorem above, we can use a time-like translation vector, i. e. a translation to a point of H^+ with $s^2 - t^2 = 1$. Here, the formulae above are still valid and simplify under the assumptions that $(s^2 - t^2)^2 - 4(s^2 - t^2) = -3 \in \mathbb{F}_q^\times$ is a square.

2. Note that the symmetries of the intersection may not leave the degenerate quadrics Q_λ invariant, rather they leave the set $\{Q_\lambda \mid \lambda \text{ is an eigenvalue of } -MN^{-1}\}$ invariant, i. e. can map one Q_λ to another Q_μ. This may be also a characterization of the automorphism group, but we did not inspect this further.

3. The above theorem can also be adapted to the case of H^- instead of H^+ since in two dimensions these two quadrics are related by a change of the first two coordinates. With this trick, we find the same result as for the case of H^+ with slightly changed assumptions.

4. The methods shown in this theorem can also be used to find the symmetry group of the intersection of H^+ with $T \triangleright H^-$, i. e. a mixing of time-like and space-like neighbours. Under the right assumptions, we also find that in $n = 2$ every possible permutation of the points of this intersection is possible. Note that the points at infinity are the same for H^+ and H^-.

For the high-dimensional cases $n > 2$, we first note that the quadrics $H_d^+ \subseteq \mathbb{F}_q\mathrm{P}^d$ for $d < n$ can be embedded into the quadric $H_n \subseteq \mathbb{F}_q\mathrm{P}^n$ as points at infinity. For example, a point $[x : y : z] \in H_2^+$, i. e. $-x^2 + y^2 + z^2 = 0$ also lies in H_3^+ as the point $[x : y : z : 0]$ since $-x^2 + y^2 + z^2 + 0^2 = -x^2 + y^2 + z^2 = 0$ by assumption. Thus, we can view H_d^+ as a subset of H_n^+ by this embedding. However, these 'sub-quadrics' all lie in the hyperplane at infinity.

Since points at infinity of H^+ lie trivially in the intersection of the quadric with the translated quadric $T \triangleright H^+$, there are at least $|H_{n-1}^+|$ points in the intersection. Hence, we conclude the following.

Proposition 3.3.7. Let $\mathbb{F}_q$ be a finite field of odd order $q \in \mathbb{N}$ with -1 not a square in $\mathbb{F}_q$, and let $Q, R \subseteq \mathbb{F}_q\mathrm{P}^n$, $n > 2, q > n$ be non-degenerate quadrics with $R = H^+$ corresponding to the symmetric representation matrix $\eta^+ \in \mathrm{Mat}((n+1) \times (n+1), \mathbb{F}_q)$ and $Q = T \triangleright H^+$ with $T \in \mathrm{PGL}(n+1, \mathbb{F}_q)$ representing a translation with respect to $H_\infty = \{x \in \mathbb{F}_q\mathrm{P}^n \mid x_n = 0\}$ with translation vector $t \in \mathbb{F}_q^n$ with $t^T \eta t \neq 0$.

Then, the automorphism group $\mathrm{Aut}(Q \cap R)$ of the intersection $Q \cap R$ is isomorphic to a strict subgroup $G \subsetneq S_{|Q \cap R|}$ of the symmetric group $S_{|Q \cap R|}$.

Proof. As noted above, the intersection of H^+ with $T \triangleright H^+$ includes at least the points of H_d^+ for $d < n$ since these can be embedded into H^+ as points at infinity. Therefore, at least $|H_{n-1}^+|$ points at infinity lie in the intersection. Using the result from 2.5.13 for even $n-1$, we find that $|H_{n-1}^+| = q^{n-2} + q^{n-3} + \ldots + q + 1$, or for even n we find $|H_{n-2}^+| = q^{n-3} + q^{n-4} + \ldots + q + 1$. Thus, we conclude that $|(Q \cap R) \cap H_\infty| > n+1$ for $q > n$. Hence, we cannot get arbitrary permutations of the points in the intersection since there is at least one case of $n+2$ points which do not form a projective frame since there are $n+1$ points in the hyperplane at infinity such that these $n+2$ points do not form a projective basis. This means that there is no projectivity which permutes these $n+2$ points with any $n+2$ points which do form a projective basis, which then leads to a strict subgroup of the symmetric group $S_{|Q \cap R|}$. □

Since we are primarily interested in four-dimensional spacetimes, a more detailed analysis is required for concrete examples in order to find the symmetry group of the intersection. One could also think about whether there exists a transformation $S \in \mathrm{PGL}(n+1, \mathbb{F}_q)$ such that a specific group G would serve as the automorphism group of the intersection $H^+ \cap (S \triangleright H^+)$. We did not consider this because other attempts seemed to be more promising at his point and did not require this much creativity which would be beyond the scope of this thesis. Note however that in our usual convention, every quadric field can be brought into this form at two different points. Additionally, we also see that such automorphisms may either leave the two planes of intersection invariant, respectively, or interchange points of one with points of the other. Hereinafter, we will consider the former case.

3.3.2 Symmetries of the Hyperplane of Intersection

In order to find more concrete examples of symmetry groups which may lead to gauge groups, we will now investigate projectivities which leave the hyperplane of intersection, i. e. in our notation H, of two quadrics which are translated with respect to each other invariant. Furthermore, we will most of the time stick to the two-dimensional case for computational purposes and use the special translation vector $t = (1, 0, \dots)^T \in \mathbb{F}_q^n$ which can be regarded as an undistorted direction of time, i. e. a translation to the most trivial time-like neighbour. Note that as seen by considerations earlier, this hyperplane of intersection does not contain any points at infinity, i. e. these are points of the affine subspace.

We will start with $H^+ \subseteq \mathbb{F}_q\mathrm{P}^n$ at the point $c = [0 : \ldots : 0 : 1] \in \mathbb{F}_q\mathrm{P}^n$. This will be the centre of this quadric. The quadric $T_t \triangleright H^+$ translated with respect to the affine space G_n with $t = (1, 0, \dots)^T \in \mathbb{F}_q^n$ then has the centre $c_t = T_t c = [1 : 0 : \ldots : 0 : 1]$ which fulfils the condition $c_t^T \eta^+ c_t = -1^2 + 1^2 = 0$. According to (3.33), the normal vector $h \in \mathbb{F}_q^n$ of the hyperplane H of intersection in the affine space is then given by $h = (1, 0, \dots, 0, -\frac{1}{2})^T \in \mathbb{F}_q^{n+1}$.

Clearly, a projectivity $M \in \mathrm{PGL}(n+1, \mathbb{F}_q)$ leaves the hyperplane invariant if we have $h^T M^{-1} = h^T$ or, equivalently, $h^T M = h^T$, both in the projective sense, since M is invertible and $Mx \in H$ for $x \in H$ if $h^T M x = h^T M^{-1} M x = h^T x = 0$. Using a block decomposition, we can write this as

$$h^T M = \begin{pmatrix} 1 & 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} a & b^T & c' \\ d & E & f \\ g & j^T & k \end{pmatrix} = \begin{pmatrix} a - \frac{g}{2} & b^T - \frac{1}{2} j^T & c' - \frac{k}{2} \end{pmatrix} \stackrel{!}{=} \lambda \begin{pmatrix} 1 & 0 & -\frac{1}{2} \end{pmatrix} \quad (3.36)$$

with $\lambda \in \mathbb{F}_q^\times$, $a, c', g, k \in \mathbb{F}_q$, $b, d, f, j \in \mathbb{F}_q^{n-1}$ and $E \in \mathrm{Mat}((n-1) \times (n-1), \mathbb{F}_q)$. This leads to the conditions $2b = j$, $g = 2(a - \lambda)$ and $k = 2c' + \lambda$, resulting in

$$M = \begin{pmatrix} a & b^T & c' \\ d & E & f \\ 2a - 2\lambda & 2b^T & 2c' + \lambda \end{pmatrix}. \quad (3.37)$$

Additionally, we want to impose the condition for M projectively preserving the centres c and c_t, respectively, such that this dynamical information remains unchanged. The projective condition $Mc = \mu c$ leads to $c' = f = 0$ and $2c' + \lambda = \lambda \stackrel{!}{=} \mu$ for some $\mu \in \mathbb{F}_q^\times$ to incorporate the projective equivalence. The second projective condition $Mc_t = \gamma c_t$

imposes the additional conditions $a = \gamma$, $d = 0$ and $2(a - \lambda) + \lambda = 2\gamma - \lambda \overset{!}{=} \gamma$ for $\gamma \in \mathbb{F}_q^\times$. This yields $\gamma = \lambda$ and

$$M = \begin{pmatrix} \lambda & b^T & 0 \\ 0 & E & 0 \\ 0 & 2b^T & \lambda \end{pmatrix}.$$

Dividing by λ and renaming $\frac{x}{\lambda} \to x$, we find that a projectivity which preserves the hyperplane of intersection H and the centres c and c_t of H^+ and $T_t \rhd H^+$, respectively, is of the form

$$M = \begin{pmatrix} 1 & b^T & 0 \\ 0 & E & 0 \\ 0 & 2b^T & 1 \end{pmatrix} \tag{3.38}$$

with $b \in \mathbb{F}_q^{n-1}$ and $E \in \mathrm{GL}(n-1, \mathbb{F}_q)$ such that M is invertible.

The resulting group has an interesting structure if its elements are transposed.

Proposition 3.3.8. Let $\mathbb{K}$ be a field with $\mathrm{char}(\mathbb{K}) \neq 2$ and $n \in \mathbb{N} \setminus \{1, 2\}$.

The group

$$G = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ b & E & 2b \\ 0 & 0 & 1 \end{pmatrix} \mid b \in \mathbb{K}^{n-1}, E \in \mathrm{GL}(n-1, \mathbb{K}) \right\} \subseteq \mathrm{GL}(n+1, \mathbb{K})$$

is isomorphic to the semidirect product $\mathbb{K}^{n-1} \rtimes \mathrm{GL}(n-1, \mathbb{K})$ with the natural action of $\mathrm{GL}(n-1, \mathbb{K})$ on $\mathbb{K}^{n-1}$.

Proof. At first, we need to show that G indeed forms a group and in particular a subgroup of $\mathrm{GL}(n+1, \mathbb{K})$. For $b = 0 \in \mathbb{K}^{n-1}$ and $E = I_{n-1} \in \mathrm{GL}(n-1, \mathbb{K})$, we find that $I_{n+1} \in G$. Furthermore, for $b, b' \in \mathbb{K}^{n-1}$ and $E, E' \in \mathrm{GL}(n-1, \mathbb{K})$, we find that

$$\begin{pmatrix} 1 & 0 & 0 \\ b & E & 2b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ b' & E' & 2b' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ b + Eb' & EE' & 2(b + Eb') \\ 0 & 0 & 1 \end{pmatrix} \in G$$

since $EE' \in \mathrm{GL}(n-1, \mathbb{K})$ and $Eb' \in \mathbb{K}^{n-1}$ such that $b + Eb' \in \mathbb{K}^{n-1}$. Thus, $G \subseteq \mathrm{GL}(n+1, \mathbb{K})$ is a subgroup.

The isomorphism between G and $\mathbb{K}^{n-1} \rtimes \mathrm{GL}(n-1, \mathbb{K})$ is then given by

$$\begin{pmatrix} 1 & 0 & 0 \\ b & E^T & 2b^T \\ 0 & 0 & 1 \end{pmatrix} \mapsto (b, E)$$

with the group structure on $\mathbb{K}^{n-1} \rtimes \mathrm{GL}(n-1, \mathbb{K})$ given by $(b, E)(b', E') = (b + Eb', EE')$ such that this map is obviously a group homomorphism and bijective. □

Since transposition of invertible matrices is a group anti-automorphism because it changes the order of multiplication, the symmetry group of the hyperplane H and the centres c and c_t is anti-isomorphic to the semidirect product $\mathbb{F}_q^{n-1} \rtimes \mathrm{GL}(n-1, \mathbb{F}_q)$.

Remark 3.3.9.

1. If we additionally demand an invariance of H^+, we find that the elements of this group are given by matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with $E^T E = I_{n-1}$. This is then clearly isomorphic to the group $\mathrm{O}(n-1, \mathbb{F}_q)$ of orthogonal matrices. Note that these also preserve the hyperplane at infinity.

2. Note that these symmetries of the hyperplane of intersection do not leave the whole intersection invariant since the hyperplane at infinity is mapped to a different hyperplane for $b \neq 0$.

3. We have already seen that a projective space can be decomposed into smaller projective spaces, consecutively. Thus, in all these subspaces we can choose a hyperplane at infinity which could then be required to be invariant such that this decomposition is preserved. There would then be a tower of different symmetry groups which are all probably related to some orthogonal or general linear group. With that, there is the idea to mimic the structure of the gauge group $\mathrm{SU}(3) \times \mathrm{SU}(2) \times U(1)$ in this descending order. Up to now, there is no valid construction of how these would arise and employed in the Finite Projective Physics framework. We will come back to this idea later.

The case $n = 2$ and $q = p$, p prime, is in particular interesting in the algebraic sense since here the group G can be found to be generated by two elements.

Lemma 3.3.10. Let $\mathbb{F}_p$ be a finite field of odd prime order $p \in \mathbb{N}$.

The group $\mathbb{F}_p \rtimes \mathbb{F}_q^\times$ with group multiplication $(a, g)(b, h) = (a + gb, gh)$ is generated by $(1, 1)$ and $(0, b)$ with $b \in \mathbb{F}_p^\times$ a generator of $\mathbb{F}_p^\times$.

Proof. The group $\mathbb{F}_p$ is generated be the element $1 \in \mathbb{F}_p$ and the group $\mathbb{F}_p^\times$ is also cyclic with generator $b \in \mathbb{F}_p$. At first, we note that the group $\mathbb{F}_p$ appears as a subgroup of $\mathbb{F}_p \rtimes \mathbb{F}_p^\times$ in the form of $\{(n, 1) \mid n \in \mathbb{F}_p\}$ since $(n, 1)(n', 1) = (n + n', 1)$ for $n, n' \in \mathbb{F}_p$. The group $\mathbb{F}_p^\times$ also appears as the subgroup $\{(0, g) \mid g \in \mathbb{F}_p^\times\} \subseteq \mathbb{F}_p \rtimes \mathbb{F}_p^\times$ because of $(0, g)(0, g') = (0, gg')$ for $g, g' \in \mathbb{F}_p^\times$. Thus, elements of these subgroups can be generated in the following form: $(n, 1) = (1, 1)^{\tilde{n}}$ for any element $\tilde{n}$ of the equivalence class of n, and $(0, g) = (0, m)^r$ with $m^r = g$.

We also note that $(n, 1)(0, g) = (n, g) = (1, 1)^{\tilde{n}}(0, m)^r$ for $n \in \mathbb{F}_q$ and $g \in \mathbb{F}_q^\times$ such that any element $(n, g) \in \mathbb{F}_p \rtimes \mathbb{F}_p^\times$ can be written as a product of $(0, m)$ and $(1, 1)$. We conclude that $\mathbb{F}_p \times \mathbb{F}_p^\times = \langle (0, m), (1, 1) \rangle$ with $(1, 1)^p = (0, m)^{p-1} = (0, 1)$. ☐

Staying in two dimensions for computational reasons, we want to look at the symmetry group G of the hyperplane of intersection in another way. We want to find a decomposition of an element $M \in G$ of the form

$$M = M_0 M_1 = M_0 T_t M_0' T_t^{-1} \tag{3.39}$$

such that M_0, M_0' leave the direction of time, i. e. $(1,0,0)^T$, invariant, M_0, M_1 leave the hyperplane H invariant and the product $M_0 M_1 = M$ leaves the centre c invariant.

This can be visualized by thinking about M_0 and M_1 as 'rotating' the space around c and c_t, respectively, such that the centre c stays invariant and that the direction of time remains unchanged. M_0' can be thought of as leaving the hyperplane with normal vector $T_t^T h$ invariant since $h^T T_t M_0' = h^T T_t$ because of $h^T T_t M_0' T_t^{-1} = h^T$. Here, we find that $h^T T_t = (1, 0, \frac{1}{2})$. Note that this could also be easily generalized to higher dimensions.

Analysing these conditions as before leads to matrices of the form

$$M_0 = \begin{pmatrix} 1 & b & g \\ 0 & e & f \\ 0 & 2b & 2g+1 \end{pmatrix} \tag{3.40}$$

with $g, b, f \in \mathbb{F}_q$ and $e \in \mathbb{F}_q^\times$ and

$$M_0' = \begin{pmatrix} 1 & b' & g' \\ 0 & e' & f' \\ 0 & -2b' & -2g'+1 \end{pmatrix} \tag{3.41}$$

with $g', b', f' \in \mathbb{F}_q$ and $e' \in \mathbb{F}_q^\times$ such that

$$M_1 = T M_0' T^{-1} = \begin{pmatrix} 1 & -b' & -g' \\ 0 & e' & f' \\ 0 & -2b' & -2g'+1 \end{pmatrix} \tag{3.42}$$

which is of the form of M_0.

Multiplying M_0 and M_1 and requiring $Mc = c$ yields

$$M = M_0 M_1 = \begin{pmatrix} 1 & be' - b' - 2gb' & 0 \\ 0 & ee' - 2fb' & 0 \\ 0 & 2be' - 2b' - 4gb' & 1 \end{pmatrix}$$

with the conditions $g - g' + bf' - 2gg' = f + ef' - 2fg' = 0$. This means that $M \in G$ and most likely, every element of G can be written in such a form, but we do not have a proper proof of this claim.

Clearly, working with this parametrization is a lot harder than the one used before. Nevertheless, it opens up a new way of looking at this symmetry group. Essentially, it is composed of one transformation around the centre c of H^+ and one around the centre c_t of $T_t \triangleright H^+$, both leaving a certain hyperplane invariant, in such a way that the centre of the first quadric and the direction to the second one is not changed. These extra conditions are essential since those prototype gauge transformations should not change the direction of time, i. e. the direction of causality, even though this could be made a lot more general, and not change the dynamical information which is stored in the centre of a quadric since the centre immensely effects a particle's trajectory by the principle of inertia. Essentially, these are transformations between two neighbouring points which effect all non-essential information whereby the hyperplane of intersection in the affine subspace between the two quadrics is essential in the sense that the points of this hyperplane are considered (virtual) intermediate steps between two neighbours in the affine subspace.

Note that up to now we did not impose any conditions on the transformations around c and c_t themselves besides a compatibility condition regarding both transformations and one regarding different hyperplanes which are connected to the hyperplane of intersection. One possibility would be to mimic a rotation which preserves the quadrics at the two different points, respectively.

In order to properly compute this, we make a change of basis of the form

$$M_0 = L\tilde{M_0}L^{-1} \tag{3.43}$$

such that L preserves the quadric H^+ as well, i. e. is a Lorentz transformation, leaves the $e_1 = (0,1,0)^T$-axis invariant and maps the hyperplane associated to the normal vector $\tilde{h} = (1,0,0)^T$ to the hyperplane H. $\tilde{M_0}$ then should leave the hyperplane with normal vector $\tilde{h}$ and the quadric H^+ invariant, respectively. The matrix $\tilde{M_0}$ is then of the form

$$\tilde{M_0} = \begin{pmatrix} 1 & 0 \\ 0 & O \end{pmatrix} \tag{3.44}$$

with $O^T O = I_2$. These orthogonal matrices can be found to be of the form

$$O = \begin{pmatrix} c & -s \\ \pm s & \pm c \end{pmatrix} \tag{3.45}$$

with $c, s \in \mathbb{F}_q$ such that $c^2 + s^2 = 1$. Hereinafter, we will choose the upper, +-case.

In formulae, we require $L \in \mathrm{PGL}(3, \mathbb{F}_q)$ to fulfil the equations $L^T \eta^+ L = \eta^+$, $Le_1 = e_1$ and $\tilde{h}^T L^{-1} = h^T$. Adapting the results of [14] which gives the structure of the Lorentz group in the case of $Le_2 = e_2$, we find that L is of the form

$$L = \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ \pm b & 0 & \pm a \end{pmatrix} \tag{3.46}$$

with $a^2 - b^2 = 1$ for $a, b \in \mathbb{F}_q$ and with matching signs in the bottom row. Its inverse is then given by

$$L^{-1} = \begin{pmatrix} a & 0 & \mp b \\ 0 & 1 & 0 \\ -b & 0 & \pm a \end{pmatrix}.$$

The projective condition $\tilde{h}^T L^{-1} = h$ then leads to

$$\tilde{h}^T L^{-1} = \begin{pmatrix} a & 0 & \mp b \end{pmatrix} \stackrel{!}{=} \mu \begin{pmatrix} 1 & 0 & -\frac{1}{2} \end{pmatrix}$$

with $\mu \in \mathbb{F}_q^\times$. With the additional condition $a^2 - b^2 = 1$, we find $\mu^2 - \frac{\mu^2}{4} = \frac{3\mu^2}{4} \stackrel{!}{=} 1$. This yields $\mu = \pm\frac{2}{\sqrt{3}}$ leading to $a = \pm\frac{2}{\sqrt{3}}$ and $b = \pm\frac{1}{\sqrt{3}}$ where the signs also depend on the sign in L.

Similarly, we make the ansatz $M_0' = L^{-1}\tilde{M}_0' L$ with $\tilde{M}_0'$ of a similar form as $\tilde{M}_0$ for the tilted transformation such that $M_1 = T_t L^{-1}\tilde{M}_0' L T_t^{-1}$ leading to

$$M = M_0 M_1 = L\tilde{M}_0 L^{-1} T_t L^{-1} \tilde{M}_0' L T_t^{-1}. \tag{3.47}$$

Note that we use L^{-1} instead of L in this transformation since $\tilde{h}^T L = (a, 0, b)$. If we choose a and b to have the same sign in (3.46), we find that $\tilde{h}^T L = h^T T_t$ up to non-zero scalar factors. This is achieved by choosing the upper case of signs in L which we will choose hereinafter.

A direct computation shows that in our case

$$L^{-1} T_t L^{-1} = \begin{pmatrix} a^2 - ab + b^2 & 0 & a^2 - 2ab \\ 0 & 1 & 0 \\ b^2 - 2ab & 0 & b^2 + a^2 - ab \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \left(T_t^{-1}\right)^T.$$

Hence, $L T_t^{-1} = T_t^T L^{-1}$ and

$$M = L\tilde{M}_0 \left(T_t^T\right)^{-1} \tilde{M}_0' T_t^T L^{-1}. \tag{3.48}$$

Thus, this is just the matrix $\tilde{M} = \tilde{M}_0 \left(T_t^T\right)^{-1} \tilde{M}_0' T_t^T$ conjugated by L.

With $v = (0, 1)^T$ and O_1, O_2 orthogonal (2×2)-matrices, $\tilde{M}$ can be computed in block form:

$$\tilde{M} = \begin{pmatrix} 1 & 0 \\ 0 & O_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -v & I_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & O_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ v & I_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ (O_1 O_2 - O_1)v & O_1 O_2 \end{pmatrix}. \tag{3.49}$$

Note that orthogonal matrices form a group under multiplication such that $O_1 O_2 = O_3$ is another orthogonal matrix. This can be used to write

$$\tilde{M} = \begin{pmatrix} 1 & 0 \\ (O_3 - O_1)v & O_3 \end{pmatrix}. \tag{3.50}$$

This could also be adapted to higher dimensions.

However, a problem arises with this structure. The product of two such matrices $\tilde{M}$ and $\tilde{M}^2$, the 2 should be viewed as an index and not a square, does not seem to be of such a form:

$$\begin{aligned} \tilde{M}\tilde{M}^2 &= \begin{pmatrix} 1 & 0 \\ (O_3 - O_1)v & O_3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ (O_3^2 - O_1^2)v & O_3^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ (O_3 O_3^2 - (O_1 - O_3 + O_3 O_1^2))v & O_3 O_3^2 \end{pmatrix} \end{aligned}$$

Clearly, $O_3 O_3^2$ is again an orthogonal matrix and $(O_3 - O_1)v = (-s_3 + s_1, c_3 - c_1)^T$ with $c_1, s_1, c_3, s_3 \in \mathbb{F}_q$ as above for O_1 and O_3, respectively. This means that the right column of $O_1 - O_3 + O_3 O_1^2$ needs to be of the form $(-s, c)^T$ with $s^2 + c^2 = 1$. It is equal to $(-s_1 + s_3 - (c_3 s_1^2 + s_3 c_1^2), c_1 - c_3 + (c_3 c_1^2 - s_3 s_1^2))^T$ which is in general not of the required form.

Thus, we conclude that our attempt did not lead to a group structure. This may be the case due to the conditions on $\tilde{M}_0$ and $\tilde{M}_0'$ which may be too restrictive. We did not follow this further since it was not a promising candidate and a bit too restrictive. However as we have seen, this and other attempts deliver rich algebraic structure, but more research is necessary to find the right candidate for the usual gauge transformation in this context. Furthermore, gauge fields and interaction carrying particles or fields are yet to be defined, but these are beyond the scope of this thesis.

3.3.3 Transformations in the Hyperplane at Infinity

We have seen that the intersection of two quadrics with one translated with respect to the other with respect to some hyperplane at infinity also consists of all the points in the hyperplane at infinity which also lie in the quadric. As we have already pointed out, these points at infinity may correspond to massless particle states which would coincide with the notion of massless gauge particles before the introduction of the Higgs mechanism in the Standard Model of Particle Physics. A mixing of points in the affine subspace with points at infinity would then be necessary to introduce masses in these states.

Recall that the gauge group of the Standard Model is given by $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$. These Lie groups are 8-, 3- and 1-dimensional, respectively. Coincidently, the algebraic groups, i. e. topological spaces such that the group operations are continuous with respect to this topology, $\mathrm{PGL}(n, \mathbb{K})$ are of dimension $n^2 - 1$ such that the groups $\mathrm{PGL}(3, \mathbb{K}), \mathrm{PGL}(2, \mathbb{K})$ are of dimension 8 and 3, respectively. Since these dimensions also give rise to the number of gauge particles, here eight gluons and $W^{\pm}$- and Z^0-bosons, these need to coincide such that the physical description stays the same.

These two groups naturally arise in a four-dimensional spacetime since the underlying projective space can consecutively be decomposed into lower dimensional projective spaces by means of choosing a hyperplane at infinity in each of these subspaces. For example, the hyperplane at infinity $H_\infty = \mathbb{F}_q\mathrm{P}^4 \setminus G_4$ is again a projective space of dimensions 3 such that $\mathrm{PGL}(4, \mathbb{F}_q)$ acts naturally on it. In this hyperplane, we can choose another hyperplane to decompose this projective space into an affine space and a hyperplane at infinity. This hyperplane will then be a projective space of dimension 2 on which $\mathrm{PGL}(3, \mathbb{F}_q)$ acts naturally. Applying this method again, we arrive at a 1-dimensional projective space corresponding to $\mathrm{PGL}(2, \mathbb{F}_q)$. Another decomposition in this way leads to just one point. This point can then be scaled by non-zero scalars leading to the same point. This could be viewed as a 1-dimensional symmetry group corresponding to $\mathrm{U}(1)$.

Remark 3.3.11. Note that the choice of affine coordinates in such a subspace could also be used to define certain particle numbers like mass or electroweak charge. The splitting with respect to some normal vector h is done by $\mathbb{F}_q\mathrm{P}^n = \{h^T x \neq 0\} \cup \{h^T x = 0\}$. In the first case, we usually take as typical representation of points in this affine space coordinates such that $h^T x = 1$, for example with $h = (0, \ldots, 1)^T$ the points with $x_n = 1$.

This could also be altered to be $h^T x = m \in \mathbb{F}_q^\times$ with m representing for example the mass or inverse mass of this particle such that $x_n = m$. Translations are then given by multiplication with the same matrices T_t with translation vector t as before. However, these would then lead to $T_t(x_0, \ldots, x_{n-1}, m)^T = (x_0 + mt_0, \ldots, x_{n-1} + mt_{n-1}, m)$. This would mean that in this approach different particles would react differently to the same translation if we were to cut only the first n coordinates in this representation. The translation could then be constructed independently from the particle numbers. Since we have different hyperplanes at our disposal in this decomposition, this could also be done similarly for the smaller subspaces with different charges like the electric charge and so on.

However, these hyperplanes are not independent of each other since they form a chain of subspaces which means that a transformation on a hyperplane of higher hierarchy also leads to a transformation in the hyperplanes below. Another problem arises from the first hyperplane. This hyperplane contains all lower hyperplanes but does not give rise to a group which can be, at least in terms of dimension, related to one of the standard gauge groups.

The hyperplane at infinity is only half of the story here. The points of the intersection of the two quadrics should also be preserved as a set by these transformations. There is also the point at infinity of the line from the predecessor to the centre of the quadric at the current event. This can be viewed as the direction of motion. Even though these could, as we have already pointed out, stay invariant, it seems also to be plausible to assume that these directions get distorted in a certain way by a transformation of the hyperplane at infinity such that this reflects the kind of interaction at hand. A gauge transformation then would be a transformation at infinity which also leaves this transformational behaviour unchanged under conjugation.

This leads to a two-fold question. On the one hand, how does the interaction mimicking transformation look like from the interaction's point of view, on the other hand, what does a given gauge transformation dictate of how these transformation associated with a certain interaction can look like. The answer to these questions, if this method indeed proves to be applicable, requires more research and is beyond the scope of this thesis.

3.3.4 Complete Quadrangles and Dynamics

The points at infinity are also interesting for another attempt of capturing the essence of a particular interaction. This is based on the concept of **complete quadrangles** and higher-dimensional analogues. These consist of four points in a plane, such that no three of which are on a common line, and of six lines connecting the different pairs of points in a projective space. These are especially of interest in the study of so-called harmonic ranges. Note that it can be proven that for any two of such complete quadrangles, there is exactly one projectivity mapping one to the other. For simplicity, we will work in two dimensions but it seems to be straightforward to generalize this to higher dimensions.

Recall that there are three essential points at each point of a trajectory of a body in our description, the event x_τ itself, its predecessor $x_{\tau-1}$ and its successor $x_{\tau+1}$. These describe completely the behaviour of the body at a given point since we only need to know where the body is in our spacetime, from where it came and whereto it goes. These three points are connected via two lines and form a plane such that this concept is admissible. Another interesting point is the centre c_τ of the quadric located at x_τ since it is directly connected to the position of the successor by the concept of inertia as explained earlier. The line connecting the centre c_τ and the event x_τ is associated to an acceleration a_τ coming from a force F_τ at x_τ. The point at infinity l_τ of this line is then essentially given by $l_\tau = [a_\tau : 0]^T$ up to some projectively irrelevant scaling factor. In two dimensions, these four points, $x_\tau, x_{\tau\pm1}$ and l_τ can then be used to define the *dynamical complete quadrangle* if the acceleration is non-zero. Otherwise, the first three points would lie on one line. The situation is similar to the one depicted in 3.1 with an additional point at infinity which cannot be drawn easily but lies on the extension of the line labelled by t.

In contrast to this dynamical quadrangle, we can also define a type of *standard complete quadrangle*. This is given by the points E_i, $i = 0, 1, 2$, representing the standard basis vectors of $\mathbb{F}_q^3$ and the point $E_3 = [1 : 1 : 1]$ which is the sum of the E_i and all the lines connecting pairs of these points. Additionally, we define the points of the intersection of these lines: $F_0 = [0 : 1 : 1]$ is the intersection of $E_0 \vee E_3$ with $E_1 \vee E_2$, $F_1 = [1 : 0 : 1]$ the intersection of $E_1 \vee E_3$ with $E_0 \vee E_2$ and $F_2 = [1 : 1 : 0]$ the intersection of $E_2 \vee E_3$ with $E_0 \vee E_1$. Note that E_3 is regarded the event in question. We now want to find a quadric Q with symmetric representation matrix M at the point E_3 such that $E_0, E_1, E_2 \in Q$ and $F_i = ME_i$ which can be viewed as the dual point of E_i with respect to Q. This matrix M can be found to be

$$M = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The challenge now is to find a suitable projectivity such that the standard complete quadrangle is mapped to the dynamical one. The point at infinity of the dynamical quadrangle then acts comparably to a force carrier. Note however, the dual points and the points of intersection which coincide in the standard quadrangle do not coincide in the dynamical one in general.

The mapping used to map one quadrangle to the other then also transforms the quadric Q to the dynamical quadric Q_τ. This gives a rather canonical description of the quadric which is based on a more geometric structure. Gauge transformations now can be considered as transformations leaving the change of the point at infinity from one dynamical quadrangle to the next one invariant, which is a transformation of the hyperplane at infinity. Vice-versa, one could think about how a particular choice of symmetry group would restrict admissible forces or force fields in this description.

This concept could then be generalized to higher dimensions as some kind of complete n-cube following the same strategy. This would lead to multiple points at infinity which may be mapped onto each other. These transformations could then be related to the other gauge types. However, this is not yet completely understood and worked out and needs more research to fully grasp.

3.3.5 Charges and Invariant Matrices

Up to now, we did only consider gauge transformations, but these are only a part of what is physically interesting. The other interesting thing are particles or fields which carry a certain charge corresponding to the interaction governed by this gauge transformation. In order to relate these charges to our description of physics, we want to incorporate them into the structure of our quadrics such that a transformation of the quadric leaves the part associated to some charge invariant. This is useful since the quadric fields of our spacetime should contain all the dynamical information needed in order to describe trajectories of particles and their interactions.

Thus, we make the ansatz for the symmetric representation matrix M of a quadric Q to be a sum of matrices which are invariant under some transformation related to a charge like isospin or electric charge, i. e.

$$M = q_1 M_{q_1} + q_2 M_{q_2} + \dots \tag{3.51}$$

with q_i a charge related to a transformation $S_i \in \mathrm{PGL}(n{+}1, \mathbb{F}_q)$ such that $(S_i^{-1})^T M_{q_i} S_i^{-1} = M_{q_i}$.

To find these matrices M_{q_i} for a given transformation S, we only have to look for the kernel of the linear map $\Phi_S \colon M \mapsto (S^{-1})^T M S^{-1} - M$ after we have fixed the scaling of S in the projective sense. Note that this fixing of a scaling factor of the transformation breaks the usual projective symmetry. A similar effect can be seen in our ansatz of M being the sum of some matrices. Since M is defined only up to a factor, the pre-factors of the M_{q_i} are only determined up to some global factor and a relative factor coming from the scale fixing of S_i. This symmetry breaking could also be used in the description of this interaction and be part of it instead of being a problem and caveat. Nevertheless, this choice seems to be quite arbitrary and has to be considered in future research.

Example 3.3.12. If we consider in two dimensions the group of matrices of the form

$$\begin{pmatrix} 1 & b & 0 \\ 0 & a & 0 \\ 0 & 2a & 1 \end{pmatrix} \text{ with } a \in \mathbb{F}_q^\times \text{ and } b \in \mathbb{F}_q,$$

we find in accordance with 3.3.10 its generators g_1, g_2 to be

$$g_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \text{ and } g_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for $\mathbb{F}_q^\times = \langle m \rangle$.

For g_1, the basis of the kernel of Φ_{g_1} can be calculated to be

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

Note that the first line of the last matrix is actually just the rescaled normal vector of hyperplane of intersection used to determine this group.

For g_2, we find the basis of the kernel of Φ_{g_2} to be

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Example 3.3.13. Another interesting example are the special orthogonal matrices of the form

$$R(c,s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix}$$

with $c^2 + s^2 = 1$. Here, the basis of the kernel of $\Phi_{R(c,s)}$ for general (c,s) is given by

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \right\}.$$

These can be interpreted as the identity on the first coordinate, the identity on the other two coordinates and an anti-symmetric matrix which can be considered an element of the Lie algebra associated with the two-dimensional orthogonal group.

This is to be expected and can be used to introduce gauge fields which usually are an element of the Lie algebra and not the group. Thus, this gives a well behaved opportunity to introduce gauge fields which correspond to a certain type of charge into the Finite Projective Physics framework.

Since there is no preferred and established way of introducing the right gauge symmetries into our spacetime, we cannot hope for the right gauge fields to appear up to now. This has to be an integral part of future research, but is beyond the scope of this thesis. Nevertheless, we wanted to convey an approach to systematically find the right fields.

4 What's Next? - Open Questions, Problems, and Ideas

At some points of this thesis, we have already hinted at some questions and problems which could not be answered yet. Most notably, these concern the question of non-constant electromagnetic fields and the condition of the transformation of one mean velocity to the next to be a Lorentz transformation, and on the other hand a suitable description and translation of a gauge symmetry to a change in the trajectory of a particle. Since electromagnetic fields and gauge fields go hand in hand, these probably cannot be answered separately. A closer look at the structure of the Lorentz force which is usually derived with variational calculus from a Lagrangian may be a good starting point. This may also raise the question of how to implement a Lagrangian or Hamiltonian to represent a model of the dynamics of a system into our framework. Furthermore, we would also like to find an analogue of Maxwell's equations for the description of the electromagnetic fields.

But there are also more conceptual questions and problems which could be considered in the future.

$\mathrm{P\Gamma L}$ instead of PGL Up to now, we used non-prime finite fields $\mathbb{F}_q$ but only used the group $\mathrm{PGL}(n+1, \mathbb{F}_q)$ for the description of our symmetry. In general, the collineation group or automorphism group of the space $\mathbb{F}_q\mathrm{P}^n$ is given by $\mathrm{P\Gamma L}(n+1, \mathbb{F}_q)$ which is the semidirect product of $\mathrm{PGL}(n+1, \mathbb{F}_q)$ with $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ with $\mathbb{F}_p$ the prime field of $\mathbb{F}_q$ and the Galois group acting on the coordinates component-wise. The Galois group of such an extension is rather simple since it is only generated by the Frobenius automorphism but the whole group $\mathrm{P\Gamma L}(n+1, \mathbb{F}_q)$ offers a richer structure than merely using $\mathrm{PGL}(n+1, \mathbb{F}_q)$.

This could also be used to define more general quadrics which are of the form $\sigma(x)^T M x = 0$ with $\sigma \in \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ and M the symmetric representation matrix of the quadric. For a quadric extension, this would be similar to the complex scalar product since in this case the Frobenius automorphism would act similarly to complex conjugation within the right choice of basis. This would be quite favourable in order to compare this theory to the usual one from Quantum Mechanics. However, this would rely on the choice of a particular field extension and a particular element of the Galois group. On the other hand, this may also lead to new features of the symmetries of such a quadric since they also have to respect this field automorphism.

Even if the form of the quadric stays unchanged, a switch to the full collineation group would lead to a more complex but also richer space to choose gauge transformations from. The different linear structures coming from the various elements of the

Galois group could also be interpreted as a gauge transformation since they are primarily showing abundances in the mathematical description when using a non-prime finite field. However, these Galois groups are cyclic which would mean that they could only be used to mimic the one-dimensional group U(1). But, this should be investigated more thoroughly.

Theory of Lines instead of Theory of Translations These much richer description would also come in handy in another possible attempt to expand this theory. In this thesis, we calculated the dynamics of a body with the help of translations with respect to some hyperplane, i. e. the forward and backward velocities. However, the dynamical information lies in the quadric itself at the point and the line connecting the predecessor with the centre of the quadric which also contains such information. Now, instead of using translations which are only defined once an affine subspace is specified we could use these lines as 'Class 1'-objects and the quadrics as 'Class 2'-objects of higher hierarchy to formulate the dynamics in our framework. The intersection of two such objects is then the key to the dynamics and trajectories.

Since PGL is not the full collineation group, the usage of the whole group PΓL would be necessary in this type of description. However, it could be found that automorphic collineations lead to certain behaviour which is unsuitable for physics. Then, we could definitely stay in the regime of PGL. But, since this has not been considered yet, we can only speculate.

Gravity Since straight lines are by Newton's laws of motion also a tool to measure force-free trajectories and since gravity is considered not to be a force but a geometrical theory where geometry and matter interact with each other, a description of gravity in the light of Finite Projective Physics as a theory of lines is at the heart of the theory. We have already indicated that gravity might also be necessary to fix the issues arising in the theory of non-constant electromagnetic fields. Thus, the relation between different lines which is given by points at infinity could be a good and well suited starting point for the theory of gravity. We can also assume that only a combination of all these theories with a non-flat spacetime would lead to proper results, hence giving us a glimpse at a possible unified theory. Up to now however, this has not been established.

The quadric field of our spacetime should be constructed not only using translations but also tilts and other transformations. This means that we have to study the behaviour of intersections of the quadric field with different lines also in these cases which may be not as easy to compute. This could also lead to some sort of 'connection' which acts on the lines and would give the next line in the process of some chain of events, i. e. the line from the event itself to the centre of the quadric located at its successor, given a quadric field and a starting line from the predecessor to the event itself which is directly given after determining the predecessor and the event itself. This could then not only be used in the theory of gravity but also in other physical situations. Essentially, given a line l_0 with $c_0 \in l_0$, this would yield a line l_1 which connects the event x_0 with the centre c_1 of the quadric $Q_1 = S \triangleright Q_0$ at its successor x_1 with the initial quadric at x_0

given by Q_0 with centre c_0 such that $x_1 \in Q_0 \cap l_0$.

Projective Space in the Context of Geometric Algebras Geometric Algebras, also known as Clifford Algebras, offer a rich tool for connections between Algebra and Geometry. There are also several uses in mathematical physics, most notably in gravity and quantum mechanics or field theories in general, e. g. [4]. In [5], it is shown how to describe a projective space within the framework of geometric algebra. Within this framework, geometrical concepts arise naturally and structural theorems can be proven with algebraic methods. In particular, these geometric algebras are interesting since they require a quadratic form on the underlying vector space. Thus, they appear in the form of spacetime algebras in the study of Minkowski spacetime. This quadratic form also gives rise to basis elements which behave like Dirac matrices if η is chosen as the quadratic form. These are essential in the study of quantum field theories.

We could imagine a space where at each point the neighbourhood is modelled by such a geometric algebra of points connected to a projective space. However, these spaces could change from point to point such that locality comes into play. At the intersection of two of those neighbourhoods, we would expect a transformational behaviour to appear just like in the case of gauge theories such that the physics remains unchanged but the mathematical description may alter. Since geometric algebras can be formulated with any field, these could also be used as a tool in the Finite Projective Physics framework. Here, we may not have a whole projective space but rather a projective space at each point with certain compatibility conditions. Since the projective spaces are also related to the quadratic form in the definition of a geometric algebra, their forms and line behaviour would also rely on the choice of a quadratic form. Yet, this is highly speculative and may not lead to a suitable structure.

Bilinear Forms constructed from $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ Up to now, the bilinear forms underlying the quadric used in our description of spacetimes were imposed externally onto this system. However, there is a canonical construction of a non-degenerate and symmetric bilinear form when using an extension field $\mathbb{F}_q$ of a prime field $\mathbb{F}_p$ with Galois group $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$. Note that this is also possible for any Galois extension. This so-called *trace form* is then also invariant with respect to the Galois group of this extension. The bilinear form is constructed as $\mathbb{F}_q \times \mathbb{F}_q \to \mathbb{F}_p, (x, y) \mapsto \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(xy)$ with $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(a) = \sum_{\sigma \in \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_q)} \sigma(a)$. This could then be used to intrinsically define the quadrics only using a special field extension and the right basis for this field extension as an $\mathbb{F}_p$-vector space.

Fibre Bundles of Neighbouring Points In our discussion of the symmetries of points at infinity or points in the plane of intersection between two translated quadrics, we only considered the transformational behaviour of the points in these hyperplanes. But, a transformation in these hyperplanes also affects other points and in particular lines in the projective space which may be unwanted. Thus, we may think about these transformations of the points of such a hyperplane as acting on a copy of these points stored in a fiber above the event in question at which the transformation should act.

Hence, we may form a type of fibre bundle with neighbouring points as fibres onto which the symmetry group act as if the points still were in the whole projective space. With that, we do not run into problems of transformations of neighbouring points also re-arranging the points and lines in the physical space. Points which are in the intersection of two quadrics are then also in the fibres at these two points which can be thought of as intersecting. It would also be necessary to label the points in these fibres such that points in the same line through the centre of the quadric have the same label and are related to the same chain of events. With this technique, it would be possible to determine all possible trajectories given a fixed (bi-)quadric field.

However, as we have already discussed, one of the main features of General Relativity is the interplay of geometry and matter such that both affect each other's behaviour. Thus, in our framework, we concluded that the presence of a body should also influence the quadric field of the spacetime and with it the structure of possible neighbouring events. This would then lead to a better understanding of how General Relativity would arise within this framework, but it is yet to be understood and requires future research.

Quantum Nature In this thesis, we treated gauge theories and electrodynamics purely classically, i. e. without any quantum theoretical background. However, in order to really adapt the Standard Model within this framework and to find the photons and the gauge bosons as quantum particle, a quantum theoretical description is necessary. There may be multiple ways to quantize within this space, but as of yet, there is no mathematically sound implementation in Finite Projective Physics. This will also have to be solved in the future.

5 Conclusion

In this thesis, we have shown that Finite Projective Physics provides a fruitful ground to describe relativistic Newtonian mechanics, in particular electrodynamics. In the case of constant electromagnetic fields, the solutions to the equations of motion, which are only described geometrically in this new framework, coincide quite well with analytical solutions respecting Special Relativity when using real numbers as the number field. In the highly relativistic regime, these rather small error seem to rise and lead to a larger gap between the two solutions. This is probably an intrinsic problem arising from the finite time step $\Delta\tau$ used in the finite description of mechanics. Further analysis will be necessary. However, we showed that special relativity is a feature of the finite projective description of mechanics since proper time is intrinsically given by the character of the event-based model with a chain of events with finite distances between them.

Another, much more conceptual problem arises in the case of non-constant electromagnetic fields. Here, additional and probably too restrictive conditions have to be imposed in order to get a Lorentz transformation between two consecutive mean velocities. However, this may also be an indication that our description is not yet complete since in this calculation we assume a flat background spacetime which is not the case in the presence of an electromagnetic field as known from General Relativity. The forward and backward velocity which also yield the mean velocity are also influenced by the choice of the hyperplane at infinity since they are merely the translation vector of a translation with respect to this hyperplane. Hence, we may conclude that in the case of non-constant electromagnetic fields a non-flat spacetime is necessary which should be constructed by means of a suitable implementation of General Relativity.

In the case of gauge transformation, we found that some of our attempts did not lead to suitable structure but others offered quite rich mathematical structure, e. g. the symmetry group of the intersection of two translated quadrics which can be thought of as intermediate steps between the two locations of these quadrics. Yet, we were not able to find a way of establishing the acting force given a certain transformation. Thus, we did not manage to prove that our attempt towards gauge transformations is right and will in the long run lead to a description of the fundamental forces as gauge theories. Still, we provided tools which can be used once the right implementation is established in the future.

We have seen that the event-based ontology employed in Finite Projective Physics offers fruitful ground on which many theories may grow in the future. Relativistic Newtonian Mechanics which has been proven to be one of these theories offers a good starting point, but future research is needed in order to fully describe our universe within this framework.

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