Investigations of Symmetry Groups in a Finite Projective Space

Master's Thesis in Physics

Presented By

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Abstract

A finite and projective geometry $\mathfrak{P}\mathbb{K}_p^d$ defined over a (d+1)-dimensional vector space is chosen as the setting for spacetime. The finite nature is endowed with the field structure of a *Galois field* F_p with an order p, where p is prime. The projective nature allows for a breakdown of the space into affine subspaces, such that an n-dimensional projective space has $\{n, n-1, \ldots, 1\}$ dimensional affine subspaces. This breakdown has been theorized to be the reason why the $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ symmetry might be seen in the particle fields in quantum field theory.

Our geometry consists of points and hyperplanes, such that an incidence relation[1] may be defined over them. The entire space is covered by a biquadric field, where a biquadric is an object of the second order. Biquadrics allow for a definition of neighborhood, order, and distance measures, and are the analogs of metric tensors in the general theory of relativity. The incidence relation itself can be thought of as an invariant, such that projective transformations, which are automorphisms of the projective space, preserve the incidence. A particular subset of these transformations are the *Lorentz transformations*, which are the automorphism group of the biquadrics. The Lorentz group also keeps the centre invariant, and is shown to consist of *boosts* or *rotations* or both, depending upon the dimension and the quadric signature considered. After a theoretic consideration of the structure of this group we also show visually the similarity between the group of rotations, which is an orthogonal group, and the normal rotations in real or complex space, such that the finite rotations can be said to be embedded in the group of rotations in complex space[2]. A similar visual approach shows that boosts and successive rotations can be axially- or point-symmetric, depending on the parameter.

While Lorentz Transformations have been discussed previously the same can not be said of *Gauge transformations*. To reach a final theory which is gauge invariant, the thesis builds upon the idea that the important considerations are twofold. The first consideration is the existence of the Local world domain. This space is a subset of the total space, such that distances (or any other second order measure) follow the same order relations as defined for a linear measure[3]. Points outside this domain, which are still part of the spacetime, must somehow be mapped back in. This mapping defines a *fiber*

[1] Incidence relations encode which point lies on which line and vice versa.

[2] This is the group U(1).

[3] This space will not be the same if one considers higher order measures.

space, such that points inside and outside successive local world domains are connected. The basic gauge invariance is then part of the degrees of freedom in chosing these mappings. Secondly, it follows from the first that one must define some sort of a mapping and look at it's symmetries. For this we look at the *intersections of quadrics*. These are called *hyperplanes of intersection* and form the basic element of our study of symmetries of this space. A similar concept is known as *interunion* and considers along with affine points also the points of intersection at infinity.

The groups defining the symmetries of the hyperplane of intersection are found and their orders and properties probed. One parameter finite subgroups, with generators, are also found for these groups. The hyperplanes of intersection are investigated for 2-dimensions, although a generalization for further parametrization is also provided for higher dimensions. Finally we also use literature to calculate orders of orthogonal, symplectic, linear, and unitary groups in the finite case. These allow us to find isomorphisms between the defined symmetries of the projective space and the groups of symmetries of the hyperplanes. In the end a particular orthogonal transformation is looked at, and shown to be central to the idea of Gauge transformations.

Contents

1 Shifting perspectives: The Finite Spacetime

There are things known and there are things unknown, and in between are the doors of perception.

Aldous Huxley, The doors of perception

The last century has proven to be pivotal for the scientific community. While change was there even before, in 1905 with Einstein's *Annus Mirabilis* papers, physics as a whole began to change. At the crux of this progression was curiosity: an ultimate search for the *Theory of Everything*. Since last century, many things have changed in our understanding of the universe. In our quest for answers we have successfuly (to a certain approximation at least) theorized about the four interactions which we believe exist. Einstein's extension of *Special Relativity*, the theory of *General relativity*, posits gravitation as an emergent phenomenon of space itself. On the other hand *Quantum Field Theory* came into the picture in the second half of the previous century, building upon the principles of quantum mechanics. While the predictions of GR on the one hand involve objects on cosmic scales (see for instance Gravitational waves [Abb16]), QFT predicts for the miniscule, where the strong and electro-weak forces become important. In recent memory, the observational evidence for the Higgs Boson, one of the fundamental particles in the standard model comes to mind [Aa12].

In his article from July 1901 [M.R01], Lord Kelvin began with the following proclamation:

> "The beauty and clearness of the dynamical theory, which asserts heat and light to be modes of motion, is at present obscured by two clouds."

This often misunderstood speech can now be put into perspective for a realization that while physics has made enormous leaps, there are still many clouds left to clear. The theory of unification is clearly a Cumulonimbus.

There are many theories which try to extend our understanding in an effort at unification. Theories beyond the standard model, string theory, and loop quantum gravity have made significant headlines. However, even such theories require extremely high or extremely low energy thresholds for predicted quantum gravity effects to become observable [Läm07]. In some cases the energy requirements can even be higher than what we may ever be able to

achieve. Such a picture then suggests one of two things. Either one of these theories is correct and now the problems are merely of measurements, or we need a fundamental shift in our understanding of physics.
Despite their differences both these theories require some sort of symmetries. Symmetries are important as part of real-world descriptions of interactions, but are also in patricular related to conserved quantities as described for finite and infinite groups by Nöther's Theorem [Leo18]. In GR for instance, *Local Lorentz Invariance* is a requirement of the theory, although the lorentz invariance is not a *global* invariance unlike in the case of special relativity. This is because the symmetry group of the Minkowski space on which the theory is based is the much bigger Poincaré group, of which the Lorentz group is a sub-group. On the other hand, QFT, which has it's origins in the quantization of Maxwell's equations of electrodynamics is a global Lorentz invariant theory. Furthermore, these are quantum Yang-Mills theories, where the quanta of fields are interpreted as particles. These fields have internal symmetries which are represented by non-abelian transformations. These are the so called *gauge transformations*, and for the case of strong, weak, and electromagnetic interactions are given by $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$. However, no such theory exists for the 4-dimensional space-time (which also takes care of the mass-gap). This problem is one of the biggest unsolved questions and is one of the 7 problems laid out in the year 2000 as part of the Milennium Prize Problems [JW00].
Most attempts at any sort of unification either build anew or use the axioms of one of the theories in an attempt to reach the other end. The latter is a more common approach but runs into problems. For instance, attempts to build a gravitation theory from a QFT run into the problem of non-renormalizability [Des00].
Attempts at using discretized space-time by using finitely many points have been made, for instance in the Causal Set Theory approach [Sur19]. This subset now includes Klaus Mecke's idea of using finite projective goemetries [Mec17] as the setting for space-time. Here finite fields, and in particular Galois fields replace the complex space of numbers. Anologous to the metric tensor, *biquadrics*, which are pairs of quadratic forms on the field are defined, which allow ordering of the set, thereby allowing descriptions of 'neighborhood'. These biquadrics might even be seen as the dynamical setting for developing a mechanics on the finite space [Mec20], but such an idea will not be discussed further in this thesis.

There are many reasons why spacetime might be modelled using finite fields. Previous efforts have been motivated for instance by the fact that singularities are not in general present in finite settings. In our work however, prime

motivations are more causal and less phenomenological in nature. They are based on an understanding that structures used in GR and QFT must arise from somewhere. Firstly, as was mentioned above GR requires the use of Riemannian and pseudo-riemannian manifolds, which are equipped with a metric. This metric is a second order symmetrix tensor, which can be represented by a matrix. The matrix representation used in GR for the 4-dimensional spacetime has the signature (number of negative and positive elements in the diagonal) as {1,3}, and in particular is written as diag$\{-1, 1, 1, 1\}$. The first coordinate is thought of as the time coordinate while the rest are spatial in nature. However, this choice is artificial, and there is no explanation as to why this form is used or required (except that it works). However one can answer this question in a finite setting, and it becomes clear that such a choice arises from the geometry itself.
In finite fields, one also has solutions to equations of the type $a^2 + b^2 = -1$. This allows for a modification of Sylvester's law of inertia, such that all quadrics in 4-dimensions are equivalent in finite geometry. Furthermore, this equivalence manifests itself also in the case of the biquadrics, where the canonical biquadrics end up with the Minkowskian (or the Euclidian) signature introduced above.

Secondly, there exists an even more fundamental question, which is also philosophical in nature. Why do we live in 4 dimensions, and in particular in 3 spatial dimensions? There is no answer in the continuous case of the $\mathbb{R}, \mathbb{C}$ fields. However, finite fields have a particular property, where in for dimensions $d >= 4$, the light cone suppresses all further dimensions. The points in the higher dimensions are such that their 'distance' to each other is 0 (like points on a light cone have 0 distance). This allows only the existence of 3 dimensions which we can see [Mec].
Thirdly, we can ask the question: *Why is there an existence of a gauge symmetry in QFT?* Why do we have a 'fiber space' which defines particles, and is disjoint from the geometry (the 'intrinsic spacetime'). Naturally the gauge symmetry also gives rise to particles. But is there any explanation in the finite field, and in particular in the finite projective geometry? The answer is yes. Firstly, there is an existence of a local domain of points in the field, and points outside the field then need to be mapped back inside. The existence of the local domain can in particular be seen as arising in the case when 'squares' are to be introduced, for instance in the case of distance. The symmetries of the local domain are then defined by the Lorentz group. However, there exists a gauge freedom in the choice of the mapping back of the points outside the local domain. In particular, this is the existence of the 'fiber space' which connects points in the local domain to the points outside of it. The symmetries

of this fiber space (for instance in 2-dimensions we have the invariance of a line) are then the gauge symmetries. The existence of an $SU(3) \times SU(2) \times U(1)$ symmetry then comes from the fact that the projective geometry in 4 dimensions can be broken down to affine subspaces which have gauge symmetries such that:

$$\mathfrak{p}\mathbb{K}_p^4 = \mathbb{K}_p^4 \cup \mathbb{K}_p^3 \cup \mathbb{K}_p^2 \cup \mathbb{K}_p^1 \quad \Longrightarrow \quad G(3) \times G(2) \times G(1)$$

Since the space-time consists now of points and lines (hyper-planes) and the object of interest is defined in terms of point-sets which relate to the biquadrics, the symmetries of these objects must be probed. In particular, the symmetries of the biquadrics are of fundamental importance, among which the orthogonal group and the Lorentz transformations in the finite case have been studied [Rei16]. Attempts at studying gauge transformations however have been fewer, with an attempt at a description being a part of the bachelor thesis of Ludwig Peschik [Pes19]. The objective of this thesis is the study of these gauge transformations, particularly in the 2-dimensional case. Since we deal with finite fields, the group theoretic representation shall not follow the same forms as those of Lie groups (of which the gauge groups discussed above are a subgroup). This thesis first introduces the notions of finite space-time and projective spaces along with biquadrics and in general translated biquadric fields which are required for the study of symmetry groups. The theory and properties of Lorentz transformations are then re-visited and a more general visual outlook of these transformations is presented. Finally an attempt is made at understanding the group of transformations which may define gauge symmetries in the finite setting.

2 Theoretical Framework

Ultimately, all moments are really one, therefore now is an eternity

David Bohm

The idea for setting up space-time in projective spaces begins with projective geometry, and in particular with the projective line in 1-dimension, and the projective plane in the 2-dimensional case. The geometry itself has its roots in the study of perspective in art [Sma09], before the mathematical setting could be created by the likes of Gino Fano [CCV13]. Here we begin with the notions of geometry and use set theoretic concepts to understand projective geometry. We then introduce coordinates and the concept of subspaces, and finally introduce the concept of groups and fields for a complete setting of our space-time.

2.1 Projective Spaces and Finite Fields

Every projective space is a type of geometry, and differs from other geometries in terms of the axioms. To understand this difference it is fundamental to go over what makes up a geometry, and then to consider why different geometries are indeed different. Here we refer to chapter 1 of [BR98].

Definition 2.1. *A Geometry $\mathscr{G}$*

A geometry is a set $\{\mathscr{O}, \mathscr{I}\}$ containing the set of objects , and a binary relation $\mathscr{I} : \mathscr{O} \times \mathscr{O} \to \mathscr{O} \times \mathscr{O}$, such that the relation is:

- *Symmetric:* $\forall P, L \in \mathscr{O} | \quad (P, L) \in \mathscr{I} \longleftrightarrow (L, P) \in \mathscr{I}$.
- *Reflexive:* $\{P, L\} \in \mathscr{I} \Longrightarrow \{L, P\} \in \mathscr{I}$

Geometries in the minimal setting are then the set $\mathscr{O}$ which consists of subsets, such as the set of points $\mathscr{P}$ and a set of lines $\mathscr{L}$. However, for geometries of more objects, for instance including planes, solids, and so on, we may have multiple objects in $\mathscr{O}$ such that $\mathscr{O} = \{\mathscr{P}, \mathscr{L}, \mathscr{H} ...\}$, and the relation $\mathscr{I}$ defines a binary relation for all these objects. To characterize these objects and therefore the geometry, we introduce the concepts of *flags* and *rank* of a geometry, so that we have a notion of distinct objects and their enumeration.

Definition 2.2. *Flag $\mathscr{F}$ and Rank $\mathscr{R}$*

For a Geometry $\mathscr{G} = \{\mathscr{O}, \mathscr{I}\}$ a flag $\mathscr{F}$ is the set of distinct disjoint subsets of $\mathscr{O}$ such that the subsets are pairwise incident with each other. A flag is said to be *maximal if there no element in $\mathscr{G}/\mathscr{F}$ such that another flag $\mathscr{F} \cup \mathscr{G}/\mathscr{F}$ exists. The flag is said to have a rank $\mathscr{R} = r$, such that we can write $\mathscr{O} = \{\mathscr{O}_1, \ldots, \mathscr{O}_i, \ldots, \mathscr{O}_r\}$. The object at position i is said to be an object of type i.*

Remark. *Points $\mathscr{P}$ are objects of type 1, and lines $\mathscr{L}$ are objects of rank 2. In general and object of rank i is made up of objects of rank $i-1$ and is therefore incident with them.*

In this thesis the central idea is based on *finite geometries* and therefore we define a finite geometry to be finite if and only if the point set $\mathscr{P} = \mathscr{O}_1$ is finite, where the notion of finiteness itself is assumed to be in the general sense[4].

With the concept of a geometry now been established, we may now ask further questions on the structure of the geometry. From everyday experience, one is already familiar with multiple geometries. Where do these distinctions stem from? It is clear then that we must focus on the relations that our objects have with each other. This relation is exactly the incidence relation we described above, and so we now move on to an axiomatic representation of this incidence relation.

Axiomatic representation: The Affine and the Projective plane

Definition 2.3. *Axioms of the Affine Plane $\mathscr{A} = \mathscr{G} = \{\mathscr{P}, \mathscr{L}, \mathscr{I}\}$ (pg.9 of [BW11])*

A1. *For any two different point $p, q \in \mathscr{P}$, there exists only one line $l \in \mathscr{L}$ that contains p and q.*
A2. *For a given line $l \in \mathscr{L}$ and a given point $p \in \mathscr{P}$ which is not on l, there exists unique line passing through p and this line does not intersect l.*[5]
A3. *There are three points on the plane such that they are not on any line at once.*

Corollary 2.1. *On an affine plane of order n, there exist n^2 points and $n(n+1)$ lines.*

Definition 2.4. *Axioms of the Projective Plane (pg.4 of [BW11])*

P1. *For any two distinct points, there is exactly one line incident with both of them.*

[4] In general finiteness refers to countability of objects in a set, such that there exists a maximum upper bound on the number of elements in the set.

[5] This is the condition of parallel lines, where parallel lines are an equivalence class.

P2. *For any two distinct lines, there is exactly one point incident with both of them.*
P3. *There are four distinct points such that no line is incident with more than two of them.*

The main difference between an affine plane and the projective plane is in the axiom of parallel lines, such that there exist no parallel lines[6]in the projective plane. This is an important distinction and results in the following.

Corollary 2.2. *The projective plane of order n has n^2+n+1 points and n^2+n+1 lines, such that every point is on $n+1$ lines, and every line has $n+1$ points.*

The fact that the number of points and lines is equal is of much importance in the construction of dual spaces later on. This duality comes directly from the fact that we have allowed all points to lie on lines and for all lines to intersect in points. The fact that the projective plane can be broken down into an affine plane and extra line can be seen from enumeration of points.

Theorem 2.1. *Deconstruction of a projective plane*

A projective plane can be written as an affine plane plus an extra line added at infinity.

Proof. We notice that $|\mathscr{P}^P|-|\mathscr{P}^A| = n+1$, and $|\mathscr{L}^P - \mathscr{L}^A| = 1$, where $|\mathscr{P}|$ represents the number of points and $|\mathscr{L}|$ represents the number of lines. The superscripts P and A represent the projective plane and affine plane respectively.
This implies that the projective axioms have one extra line and $n+1$ extra points. Since the projective plane has $n+1$ points on 1 line, we interpret this as extra line in the projective plane. In terms of affine planes, this is the space where the equivalence class of n parallel lines can be thought of as meeting. Hence we call it the line at infinity[7].

□

Having the idea of an affine plane and a projective plane, we can move on to construct projective spaces. However, we must talk more about the finiteness of the space, which is a fact of the underlying field. Since Field and later on our transformations are parts of group, we now move on to a discussion of groups and by extension of fields, where we have referred to chapter 1 of [LN97].

[6]Except for the line to itself.
[7]In higher dimensions this is replaced by a hyperplane at infinity.

2.2 Groups and Fields

Definition 2.5. *Group G*

A group is a set G together with a binary operation on G such that the following three properties hold:

1. ** is associative; that is, for any $a, b, c \in G$,*
$$a * (b * c) = (a * b) * c$$

2. *There is an identity (or unity) element e in G such that for all $a \in G$,*
$$a * e = e * a = a$$

3. *For each $a \in G$, there exists an inverse element $a^{-1} \in G$ such that*
$$a * a^{-1} = a^{-1} * a = e$$

If the group also satisfies:

4. *For all $a, b \in G$,*
$$a * b = b * a$$

then the group is called abelian (or commutative).

Definition 2.6. *Cyclic Group*

A multiplicative group G is said to be cyclic if there is an element $a \in G$ such that for any $b \in G$ there is some integerj with $b = a^j$. Such an element a is called a generator of the cyclic group, and we write $G = (a)$.

Example 2.1. *Cyclic Groups*

1. *The additive group of integers modulo n, $\mathbb{Z} \pmod{n}$ has the generator 1. The generators of the multiplicative group which is $\mathbb{Z}*/ \pmod{n} = \mathbb{Z}/ \pmod{n} - \{0\}$ depend on the prime.*

2. *The group of Integers $\mathbb{Z}$ ûnder normal addition has two generators {-1,1}.*

3. *Except for the identity group, the identity element can never be the generator of a group.*

There are many important properties associated with groups, and the study of groups is possibly among the most important studies in physics. Here we go over some of the more pertinent aspects for our work, which involve some basics properties of groups and the definition of an equivalence relation on it.

Definition 2.7. *Finite Group*

A group is called finite (resp. infinite) if it contains finitely (resp. infinitely) many elements. The number of elements in a finite group is called its order. We shall write $|G|$ for the order of the finite group G.

Definition 2.8. *Subgroup H*

A subset H of the group G is a subgroup of G if H is itself a group with respect to the operation of G. Subgroups of G other than the trivial subgroups $\{e\}$ and G itself are called nontrivial subgroups of G.

The subgroup of a group partitions the group into smaller sets, which are called the *co-sets*.

Theorem 2.2. *Lagrange's theorem for finite groups (see for instance [Rot01])*

Let $H \subset G$, then $\exists n \in \mathbb{Z}^+ : |G| = n|H|$. n is called the index of H in G, and defines the number co-sets of H in G.

Corollary 2.3. *Let G be a group of prime order[8]. Then G has no subgroups[9]and hence is cyclic.*

Corollary 2.4. *Any finite group of prime order is isomorphic to the group $\mathbb{Z} \pmod p$.*

Example 2.2.

1. *The additive group of integers modulo 4 has 4 elements $\mathbb{Z} \pmod 4 = \{0, 1, 2, 3\}$. The subgroup $\{0, 2\}$ has 2 elements and divides the group into $4/2 = 2$ subsets, itself and the other subset $\{1, 3\}$.*

2. *The group of Integers modulo 5, $\mathbb{Z} \pmod 5$ is a group of prime order and has no sub-group.*

3. *The group of translations in the 1-dimensional projective line with a homomorphism defined from the translations to the group of 2*2 matrices has the form $T = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. The group has p elements, is isomorphic to $\mathbb{Z} \pmod p$ and therefore cyclic, with the generator $= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.*

The existence of groups allows us to study structures and sub-structures. Of importance in this study is the fact that one can define *mappings* between

[8]This means that the order of the group is a prime number.

[9]Since in this case, there is no division of the prime order into seperate sets.

different groups, such that the study of multiple groups can be condensed. Such mappings are called *homomorphisms*.

Definition 2.9. *Homomorphism* Φ

A mapping $\Phi : G \to H$ from the group G to the group H is called a homomorphism of G into H if Φ preserves the operation of G. If $*$ and $\cdot$ are the operations of G and H repectively, then we have,

$$\forall (a,b) \in G, \quad \Phi(a * b) = \Phi(a) \cdot \Phi(b)$$

If Φ is onto H, then it is called an ***epimorphism****. A mapping from G to itself is called an* ***endomorphism****. Importantly, a mapping which is also one-to-one is called an* ***isomorphism****, and the two groups are said to be* ***isomorphic****. Finally, an isomorphism which is an endomorphism is called an* ***automorphism****.*

Of importance are those elements which are mapped to the identity element.

Definition 2.10. *Kernel*

The kernel of a homomorphism $\Phi : G \to H$ is the set:

$$ker_\Phi = \{a \in G : \Phi(a) = e'\}$$

where e' is the identity element of the group H.

Remark. *The kernel of a group is always a subgroup of the group and has the special property that if $a \in G, b \in ker_\Phi$ then $aba^{-1} \in ker_\Phi$.*

This allows us to define this special subgroup.

Definition 2.11. *Normal subgroup N*

A subgroup N of the group G is called the normal subgroup if $\forall a \in G, n \in N : ana^{-1} \in N$.

Remark. *All subgroups of an abelian group are normal subgroups. The subgroups of any abelian subgroup of a non-abelian group are normal with respect to the abelian subgroup only.*

Normal subgroups are the most important subgroups of any group. These and only these can be used to define *factor groups* of any group.

Theorem 2.3. *If H is a normal subgroup of G, then the set of (left) cosests of G modulo H forms a group. The group operation is (aH)(bH)=(ab)H.*

Definition 2.12. *Factor Group*

For the normal subgroup H of the group G, the group of left cosets of H formed with the operation defined above is called the factor or the quotient group, and is denoted by G/H.

Remark. *For finite groups, we have:*

$$|G/H| = \frac{|G|}{|H|} \tag{2.1}$$

Theorem 2.4. *Let $\Psi : G \to \Psi(G) = G_1$ be a homomorphism of a group G onto a group G_1. Then ker_Ψ is a normal subgroup of G, and the group G_1 is isomorphic to the factor group G/ker_Ψ. Conversely, if H is a normal subgroup of G, then the mapping $\Phi : G \to G/H$ defined by $\Phi(a) = aH, fora \in G$ is a homomorphism of G onto G/H with $ker_\Phi = H$.*

One generalizes equivalence relations on groups, since equivalence relations help partition groups into lesser elements.

Definition 2.13. *Equivalence Relation*

In general, a subset R of $S \times S$ is called an equivalence relation on a set S if it has the following three properties:

1. *$(s,s) \in R, \forall s \in S$ (reflexivity).*
2. *If $(s,t) \in R$, then $(t,s) \in R$ (symmetry).*
3. *If $(s,t),(t,u) \in R$, then $(s,u) \in R$ (transitivity).*

Definition 2.14. *Ring R*

A ring $(R,+,\cdot)$ is a set R, together with two binary operations, denoted by $+ and \cdot$, such that:

1. *R is an abelian group with respect to +.*
2. *$\cdot$ is associative; $\forall a,b,c \in R : a \cdot (b \cdot c) = (a \cdot b) \cdot c$*
3. *The distributive laws hold; that is, $\forall a,b,c \in R$:*
   $a \cdot (b+c) = a \cdot b + a \cdot c$ *and* $(b+c) \cdot a = b \cdot a + c \cdot a$

The concept of a ring becomes important due to the introduction of two operations as opposed to just one. From the concept of a ring, one can come to a more specific form, which is that of a field. More generally:

Definition 2.15. *Properties of Ring R*

1. *A ring is called a ring with identity if the ring has a multiplicative identity; that is, $\exists e \in R$, such that $\forall a \in R, a \cdot e = e \cdot a = a$.*
2. *A ring is called commutative if $\cdot$ is commutative.*
3. *A ring is called an integral domain if it is a commutative ring with identity $e \neq 0$ in which $ab = 0$ implies $a = 0$ or $b = 0$.*
4. *A ring is called a division ring (or skew field) if the nonzero elements of R form a group under $\cdot$.*
5. *A commutative division ring is called a field.*

The requirement of an integral domain is important since it is the generalization of integers on the real number line. This allows a more real world mapping of properties. It can be shown trivially then that every finite integral domain is also a field

Example 2.3.
1. *Let R be any abelian group with group operation +. Define ab=0 $\forall a, b \in R$. Then R is a ring.*
2. *The integers $\mathbb{Z}$ form an integral domain*[10]*and not a field.*
3. *The set of all 2×2 matrices with real numbers as entries forms a non-commutative ring with identity with respect to matrix addition and multiplication.*
4. *The ring formed by the equivalence (residue) classes of the integers modulo the principal ideal*[11] *generated by a prime p, $\mathbb{Z}/(p)$ is a field*[12]*.*

The last example underlies the field we shall be working with.

Galois Fields

In theory one can define a projective geometry or any other sort of geometry by choosing the elements from a field. The fields $\mathbb{R}$ and $\mathbb{C}$ are usually

[10]Again, the idea of integral domains is actually a generalization of the integers.
[11]An ideal is the equivalent of the normal subgroup for a ring.
[12]Since this ring is finite and is also an integral domain.

used to define space-times. Here, however, we work specifically with Galois Fields (pg.15 of [LN97]).

Definition 2.16. *Galois Fields $\boldsymbol{F}_p$*

For a prime p, let $\boldsymbol{F}_p$ be the set $\{0, 1, \ldots, p-1\}$ of integers. Let $\phi : \mathbb{Z}/(p) \to \boldsymbol{F}_p$ be the mapping defined by $\phi([a]) = a$ for $a = 0, 1, \ldots, p-1$. Then $\boldsymbol{F}_p$, with the field structure induced by ϕ, is a finite field, called the Galois Field of order p.

The mapping ϕ is an isomorphism and allows the definition in terms of $\mathbb{Z}/p$. The field $\mathbf{F}_p$ then has the zero element 0, the identity element as 1, and the elements of $\mathbf{F}_p$ follow arithmentic operations $+, *$ defined over *Integers modulo p*.

The usage of *Galois Fields* as the underlying structure has two specific advantages. Firstly, in the large number or classical limit, working with $\mathbf{F}_p$ is the same as working with $\mathbb{Z}$. An example of this, talking about the *Local World Domain* will be shown in the next section. The second advantage is in the modelling of possible quantum effects, which may arise naturally in such a field due to the periodic nature of the field for smaller primes. An example of this will be discussed in the section of *Gauge Transformations*.

Projective Spaces over Galois Fields

We now move to a more analytic expression of projective spaces. The introduction of Fields allow us to *coordinatize* the space, and to define vector spaces over them.

Definition 2.17. *Projective Space $PG(n, K)$* (see pg.29 of [Hir79])

Let $V = V(n+1, k)$ be an $(n+1)$–dimensional vector space over the field K, without the origin. We consider the equivalence relation on the points of $V \setminus \{0\}$ whose equivalence classes are the one-dimensional subspaces of V without the origin. Formally,

$$\textit{If } X, Y \in V \setminus \{0\}, \textit{where } X = (x_0, x_1 \ldots, x_n), Y = (y_0, y_1, \ldots, y_n) \textit{ in some basis}$$

$$\textit{then, } X \equiv Y \textit{ if, for some } \lambda \in K, y_i = \lambda x_i \forall i.$$

Then the set of equivalence classes is the n-dimensional projective space over the field K, and we denote it by $PG(n, K)$.

When the Field is the Galois Field, we denote the Projective space as $PG(n, q)$, where q is the *characteristic of the Galois Field*. Sometimes the d-dimensional projective space is also denoted as $\mathfrak{P}\mathbb{K}_p^d$.

The elements of $PG(n,K)$ are called *points*. If the point $\mathbf{P(X)}$ is the equivalence class of the vector X, then we say that X is a vector representing $\mathbf{P(X)}$. Therefore, λX, $\forall \lambda \in K$ also represents $\mathbf{P(X)}$. For elements other than *points*, like *lines*, *planes*, and so on, we define sub-spaces of the projective space.

Definition 2.18. *Subspace of dimension m of $PG(n,K)$*

A subspace of dimension m of $PG(n,K)$ or an m-space is a set of points, all of whose representing vectors form (together with the origin) a subspace of dimension $m+1$ of $V(n+1,K)$. We represent the subspace as π_m.
Points are subspaces of dimension 0. 1–dimensional subspaces are called lines, while 2–dimensional are called planes. The subspaces of dimension $(n-1)$ are called hyperplanes.

If a point P lies on a hyperplane H, then P is *incident* with H, and conversely H is *incident* with P. We represent this by writing the point as X and the representational matrix of the hyperplane as A, which then follows that

$$XA = 0, \quad X \in M_{(1\times n+1)}, \ \ A \in M_{(n+1\times n-m)} \tag{2.2}$$ [13]

The *incidence* of points and hyperplanes is the most important property of the projective space, and distinguishes it from other forms of geometry. This incidence structure allows us to define mappings on it, and especially important are those mappings which keep the incidence structure invariant. We call these mapping *collineations*, since they map co-linear points to co-linear points.

Definition 2.19. *Collineation ζ*

If S and S' are two projective spaces, a collineation is a bijection $\zeta : S \to S'$, such that the incidence is preserved. Therefore,

$$\pi_r \subset \pi_s \Longrightarrow \pi_r\zeta \subset \pi_s\zeta \tag{2.3}$$

Since a *collineation* is by defintion a bijection, inverses exist, and the ones of particular interest are then the subset which we call *projectivities*.

Definition 2.20. *Projectivity Π*

A projectivity is a bijection $\Pi : S \to S'$ given by a matrix ρ. If $\boldsymbol{P}(X') = \boldsymbol{P}(X)\Pi$, then $\rho X' = X\rho$, where X,X' are the coordinate vectors of $\boldsymbol{P}(X)$ and $\boldsymbol{P}(X')$, and $\rho \in K$. Furthermore, for the projectivity $\Pi : M(\rho)$ the matrix $M(\lambda\rho)$ defines the same projectivity $\forall \lambda \in K$.

[13] $M_{i\times j}$ denotes the set of matrices of dimension $i \times j$.

Since the *projectivities* are again bijections, their matrices are non-singular. Furthermore, in this thesis we are concerned with the case when the bijection is defined from the projective space to itself. Therefore, we work with *Automorphisms* of the projective space. The *automorphisms* of the space form a group $Aut(K)$, which are represented by invertible matrices. The group of projectivities from $PG(n,K)$ to itself is denoted $PGL(n+1,K)$, where $(n+1)$ represents elements of matrices of $(n+1)\times(n+1)$ dimension.

The fundamental theorem of Projective Geometry states that for a projectivity Π such that the point set $\mathbf{P}(X)\Pi = \mathbf{P}(X)\rho$, $\mathbf{P}(X)\Pi = \mathbf{P}(X^{\sigma}\rho)$, where $\sigma \subset$ Aut$(PG(n,K))$ [HT16].

A subset of the projectivities are the so called *Affinities* which define automorphisms of the affine subspace. However, for these we must first define the affine subspace itself.

Affine Spaces and Lines at Infinity

A projective space can be thought of as the generalization of an affine space with an extra plane added. This extra plane is called the *hyperplane at infinity*, which from here on will be written as $\mathbf{H}^{\infty}$. The hyperplane at infinity for the case of $PG(2,q)$ is the line at inifinity, written as ℓ^{∞}.[14]

Definition 2.21. *Affine subspace*

An affine subspace $\mathbb{K}$ *of dimension* (n) *for the projective space* $PG(n,q)$ *is the vector space* $V(n,q)$ *including the centre. In general an affine subspace can be an* $r-$*dimensional space* $\mathbb{K}^r$ *defined by* $V(r,q)$. *The affine subspace allows us to define the projective space as:*

$$PG(n,q) = \mathbb{K}^n \cup \boldsymbol{H}^{\infty} \tag{2.4}$$

This decomposition is not only limited to the affine plane however, and one can further decompose the hyperplane at infinity into affine subspaces. This is the generalization of the two-dimensional case of a projective plane being decomposed into an affine plane and a line at infinity, where the line at infinity can be further written down as an affine line and a point at infinity. Since projective spaces are defined by 2-dimensional subsets, it is always possible to decompose a projective space into such components. Here we refer to pg.2 of [Mec17].

[14] In general a line is just a hyperplane in 2-dimensions, but we distinguish them here for future use.

Definition 2.22. *Affine subspaces of the hyperplane at infinity*

For any $PG(n,q) = \mathbb{K}^n \cup \boldsymbol{H}^\infty$, where $\boldsymbol{H}^\infty$ is a $(n-1)$ dimensional space, one has further that:

$$\boldsymbol{H}^\infty = \mathbb{K}^{n-1} \cup \mathbb{K}^{n-2} \ldots \cup \mathbb{K}^1 \tag{2.5}$$

such that the projective space can be written as:

$$PG(n,q) = \mathbb{K}^n \cup \mathbb{K}^{n-1} \cup \mathbb{K}^{n-2} \ldots \cup \mathbb{K}^1 \tag{2.6}$$

The breakdown of the projective space along with the definition of equivalence classes clearly suggests that the projective space is not the same dimension as the underlying vector space. Since we have defined equivalence classes of 1-dimensional subsets, this means that all points of the form λP are the same as the point P. This feature of projective spaces goes back to before their geometric considerations, wherein all points lying on the same line were considered to be equivalent. Therefore for any point P we have the $p-1$ points λP, which all together equivalent. This equivalence is also the reason why the vector space is considered without the $\{0\}$, since this point has no equivalent points.

The understanding of this peculiar feature allows us to define coordinates. For an $n+1$ dimensional vector space one can select any arbitrary basis of $n+1$ vectors, such that the vector space is spanned by them. One writes the points then as $P = \{p_1, p_2, \ldots, p_{n+1}\}$. However the equivalence allows us to write the affine points instead as $P \sim P_{\text{aff}} = \{\vec{p}, 1\}$, where we now have the n-dimensional vector $\vec{p}$. We then chose the centre of this affine space to be the 'centre' of the n-dimensional vector space, and so the centre becomes the point $\{0, 0, \ldots, 1\}$.

Similarly the breakdown of the hyperplane at infinity becomes clear as well, and we define the $n-1$ dimensional hyperplane at infinity with the coordinates of the form $P_{\text{inf}} = \{p, 0\}$. The breakdown of the hyperplane at infinity into lower order subspaces is then also analytically clear, since one can now write:

$$P_{\text{inf}} \rightarrow P_{\text{inf}}^{\text{aff}} = \{\vec{p}, 1, 0\}, \tag{2.7}$$

$$P_{\text{inf}} \rightarrow P_{\text{inf}}^{\text{inf}} = \{p, 0, 0\} \tag{2.8}$$

and so on till we reach the final point at infinity which is given by the coordinates $\{1, 0, 0, \ldots, 0\}$.

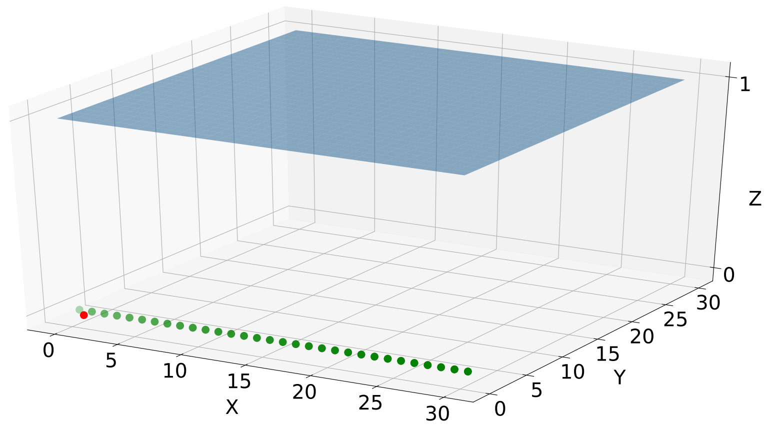


Figure 1: The breakdown into an affine plane (blue), an affine part for the line at infinity (green), and a point at infinity (red) for the 2-dimensional projective space as given by equation (2.6). Number of points are according to prime $p = 31$.

The breakdown of the two-dimensional projective space in the 3-dimensional vector space is given in figure 1.

Dual Spaces

The idea of dual spaces is central to projective geometry and allows any property of projective spaces to carry on to affine spaces (pg.31 of [Hir79]).

Definition 2.23. *Principle of Duality*

For any $S = PG(n, K)$ there exists a dual space $S*$. The dual space $S*$ is defined by interchanging the points and the hyper-planes of the space S. Formally,

$$S = \{P, H, I\} \implies S* = \{H, P, I\}$$

such that the incidence relation is invariant. Any theorem that holds for S, then also hold for $S*$.

In $PG(2, K)$ the points are dual to lines, in $PG(3, K)$ the points and the planes are dual to each other. In general for $PG(n, K)$ the subspace π_r will be dual to π_{n-r-1}. The duality allows us to define hyperplanes from points and vice versa, employing the use of a *cross product*. A similar mapping called the *dot product* further enables us to check for incidence as well [Las14].

Definition 2.24. *Dot Product* $\cdot$

Dot Product $\cdot$ *is defined as a product between points and their duals, such that:*

$$\cdot : \mathbb{P}^d\mathbb{F} \times \mathbb{P}^d\mathbb{F}^* \to \mathbb{F} : \tag{2.9}$$

$$P \cdot H = \sum_{i=1}^{d+1} p^i h_i = P^T H \tag{2.10}$$

Definition 2.25. *Cross Product*

The cross product is an operation which exists for both a projective space and it's dual, and is allows one to go from one space to the other. Therefore we define the cross product as existing for both points and hyperplanes, such that:

$$(\mathbb{P}^d\mathbb{F} \times \dots \mathbb{P}^d\mathbb{F}) \to \mathbb{P}^d\mathbb{F}^* \tag{2.11}$$

$$(\mathbb{P}^d\mathbb{F}^* \times \dots \mathbb{P}^d\mathbb{F}^*) \to \mathbb{P}^d\mathbb{F} \tag{2.12}$$

Where the operator is defined as:

$$\times : (v_1 \dots v_n) \to (v_1' \dots v_n') : \det \begin{pmatrix} v_1 \\ \vdots \\ v_n \\ \dots \\ \hat{e} \end{pmatrix}, \hat{e} = (\hat{e}_1 \dots \hat{e}_{d+1}) \tag{2.13}$$

In the projective plane, the cross product of two lines is the point at which they intersect, whereas the cross product of two points is the line on which both the points lie. For projective spaces, the definition is expanded to include higher subspaces which define hyperplanes. Similarly a dot product allows us to define an incidence relation between a point and a hyperplane.

Definition 2.26. *Condition of Incidence*

If a point $P \in H$*, then* P *is said to incident with* H*, and the condition is given as:*

$$P \cdot H = P^T H = 0 \tag{2.14}$$

The cross product then allows us to define the dual space of points in the following manner.

Definition 2.27. *Hyperplane dual to a Point*

A hyperplane H is defined as k-dimensional space which is dual to a point P_0 and is given by the cross-product of a set of points $P_0, \ldots, P_{k-1}$, where the cross-product is the determinant of the form:

$$H = \det \begin{pmatrix} P_0 \\ \vdots \\ P_{k-1} \\ \cdots \\ \hat{e} \end{pmatrix} \tag{2.15}$$

2.3 Neighborhood, Quadric, Biquadrics

Now that we have defined our space and equiped it with structure we move on to defining neighborhoods. From here on, projective spaces shall refer to n-dimensional spaces over Galois Fields, unless states otherwise. While the use of Galois Fields has many advantages in imparting minimal structure to our spacetime, the biggest drawback is the lack of an order relation on it. This means that there is no way to know what point (in general event) comes *before* or *after*. To solve this problem, a bilinear form is introduced. This is know as a *quadric* and is analogous to the *metric tensor* used in GTR.

Definition 2.28. *Bilinear Form B [Edg04]*

A Bilinear form B on $\mathbb{F}_p^n$ is a map from $\mathbb{F}_p^n \times \mathbb{F}_p^n \to \mathbb{F}_p$, which is defined by $(x, y) \to B(x, y)$ such that:

1. $B(x + x', y) = B(x, y) + B(x', y)$
2. $B(x, y + y') = B(x, y) + B(x, y')$
3. $B(\lambda x, y) = \lambda B(x, y) = B(x, \lambda y)$

Quadratic forms which we are concerned with also have the *property of symmetry* such that:

$$B(x, y) = B(y, x) \tag{2.16}$$

The property of symmetry implies that a point which has a neighbor point is also the neighbor of that point.

Definition 2.29. *Quadratic form F and Quadric* $\boldsymbol{Q}$

A quadratic form $F \in F_q[X_0, \dots, X_n]$*, where,*

$$F = \sum_{i=0}^{n} a_i X_i^2 + \sum_{i<j} a_{ij} X_i X_j, \tag{2.17}$$

The form can be represented by a matrix M_Q *such that*

$$\forall F, \exists M_Q : M_{ij} = a_{ij} \tag{2.18}$$

Then the quadric $\boldsymbol{Q}$ *is the kernel of the representation and we denote the representation by* $Q \in M_Q$*, such that given a point set* $\boldsymbol{P}_Q$

$$\forall p \in \boldsymbol{P}_Q \Longleftrightarrow p^T Q p = 0 \tag{2.19}$$

An alternative definition also exists in terms of the bilinear forms introduced above.

Definition 2.30. *A Quadratic Form Q is a map from* $\mathbb{F}_p^n \to \mathbb{F}_p$*, defined by* $x \to Q(x)$ *such that:*

1. $Q(\lambda x) = \lambda^2 Q(x), \forall \lambda \in \mathbb{F}_p$ *and* $\forall x \in \mathbb{F}_p^n$.
2. $B(x, y) = Q(x + y) - Q(x) - Q(y)$ *is bilinear. We call B the polarization of Q.*

Quadrics allow for an ordering of space, and due to being a quadratic function, one has two solutions in either direction (hence a *before* or an *after* can be defined). The canonical forms of a quadric can be found and they can be divided into parabolic, hyperbolic, and elliptic quadrics. The difference in using a finite field though is that fewer quadrics are projectively distinct than in the $\mathbb{R}$ case for instance. This is because in the case of $p \equiv 3 \pmod 4$, the equation $x^2 + y^2 = -1$ has solutions, and it is possible to change the signature of the matrix (see for instance Sylvester's law of inertia for the real case).

The number and form of the quadrics depends on the dimension, but in general one can define three canonical quadric forms (pg.7 of [Edg04]).

Theorem 2.5. *Any quadratic form over* $\mathbb{F}_p$ *is of one of the following forms:*

1. *Parabolic:* $Q_0(x) = x_1 x_2 + x_3 x_4 + \dots + x_{n-2} x_{n-1} + c x_n^2, c \in \{1, a\}$ *where* $a \in \mathbb{F}_p^{*2}$ *if p is odd and c is 1 if p is even.*
2. *Hyperbolic:* $Q_1(x) = x_1 x_2 + \dots + x_{n-1} x_n$.

3. *Elliptic:* $Q_2(x) = x_1x_2 + \ldots + x_{n-3}x_{n-2} + p(x_{n-1}, x_n)$*, where* $p(x_{n-1}, x_n)$ *is an irreducible quadratic form in 2 variables.*

The representations of these quadrics in even dimension d is given by the canonical parabolic form

$$Q_{\text{par}} = \begin{pmatrix} \mathbf{I}_d & 0 \\ 0 & 1 \end{pmatrix} \tag{2.20}$$

and in the odd dimensions $d+1$ either as the hyperbolic or the elliptic form as

$$Q_{\text{hyp}} = \begin{pmatrix} \mathbf{I}_{(d+1)} & 0 \\ 0 & 1 \end{pmatrix}, \quad Q_{\text{ell}} = \begin{pmatrix} \mathbf{I}_{(d+1)} & 0 \\ 0 & ns \end{pmatrix} \tag{2.21}$$

where ns is a non-square in a Galois field.

Lemma 2.1. *The quadric centre on a point p allows us to define a polar hyperplane. The polar hyperplane wrt a quadric Q_p centred at p is given as:*

$$H_{pol} = Q_p P$$

Furthermore, for the centre $C = \{0, 0, \ldots, 1\}^T$, the hyperplane at infinity $H^\infty = \{0, 0, \ldots, 1\}^T$ is the polar hyperplane wrt to canonical quadrics Q_{min} and Q_{euc}.

In general any line that passes through the centre will intersect the quadric in two points. However the existence of a quadric doesn't guarantee a point in every direction, meaning that we have too few points and too many lines (passing through the centre). We show the proof here as follows.

Theorem 2.6. *A quadric Q intersects only half the lines passing through its centre.*

Proof. A quadric Q has points which are solutions of $p^T Q p = 0$. These are a total of $p+1$ points. We now see that:

$$p^T Q p = 0 \Longrightarrow (-p)^T Q(-p) = 0 \tag{2.22}$$

This means that the point $-p$ is also a quadric point. Now let a line l be a line through the centre. Then this line belongs to a family of $p+1$ lines, and if the line intersects the quadric point p we have:

$$p^T l = 0, \tag{2.23}$$

$$\Longrightarrow \quad (-p)^T l = 0 \tag{2.24}$$

This further shows that a line through the centre intersects the quadric in two points p and $-p$. Therefore for $(p+1)$ points we only have $(p+1)/2$ lines actually interesecting the quadric.

□

The above case happens because in a finite field not all numbers are squares. The solution for this problem is then to add another quadric to the centre, which takes care of the non-square values. The two quadrics together are then called a biquadric (see pg.54 of [Las14]).

Definition 2.31. *Biquadric $\boldsymbol{Q}^{\pm}$*

A Biquadric is defined as the pair of matrices $\{Q^+, Q^-\}$, such that every line passing through the centre intersects the lines. The form of the two matrices is given by adding a non-square $(ns \in F_p)$ as the last element of the matrix, such that we have the Minkowski biquadric as:

$$Q^+_{min} = \begin{pmatrix} -1 & \boldsymbol{0} & 0 \\ \boldsymbol{0}^T & \boldsymbol{I} & \boldsymbol{0}^T \\ 0 & \boldsymbol{0} & 1 \end{pmatrix}, Q^-_{min} = \begin{pmatrix} -1 & \boldsymbol{0} & 0 \\ \boldsymbol{0}^T & \boldsymbol{I} & \boldsymbol{0}^T \\ 0 & \boldsymbol{0} & ns \end{pmatrix} \tag{2.25}$$

and the Euclidian biquadric as:

$$Q^+_{euc} = \begin{pmatrix} 1 & \boldsymbol{0} & 0 \\ \boldsymbol{0}^T & \boldsymbol{I} & \boldsymbol{0}^T \\ 0 & \boldsymbol{0} & 1 \end{pmatrix}, Q^-_{euc} = \begin{pmatrix} 1 & \boldsymbol{0} & 0 \\ \boldsymbol{0}^T & \boldsymbol{I} & \boldsymbol{0}^T \\ 0 & \boldsymbol{0} & ns \end{pmatrix} \tag{2.26}$$

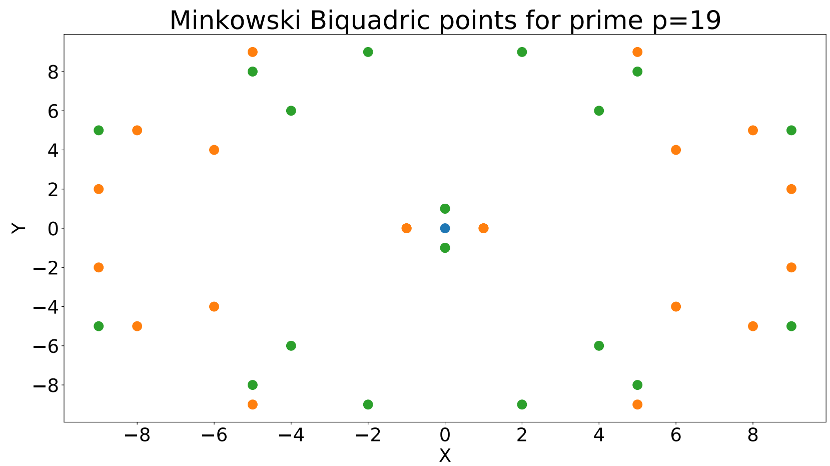
The quadric Q^+ used in this thesis will be represented by a pre-metric G, where we define it as

Definition 2.32. *The pre-metric G*

The pre-metric G is defined either as being Minkowskian G_{min} or as being Euclidian G_{euc}. The difference is naming arises due to difference in the signature of the two matrices. We have:

$$G_{min} = \begin{pmatrix} -1 & \boldsymbol{0} \\ \boldsymbol{0}^T & \boldsymbol{I} \end{pmatrix}, G_{euc} = \begin{pmatrix} 1 & \boldsymbol{0} \\ \boldsymbol{0}^T & \boldsymbol{I} \end{pmatrix} \tag{2.27}$$

The neighborhood for the biquadrics centred at {0,0,1} is shown in figure 2. One can see that there is a line intersecting the biquadric in every direction, except for the two lines (asymptotes to the hyperbola) in the Minkowski biquadric which intersect the quadric at the quadric point at infinity. In the Euclidian case this distinction is lost.



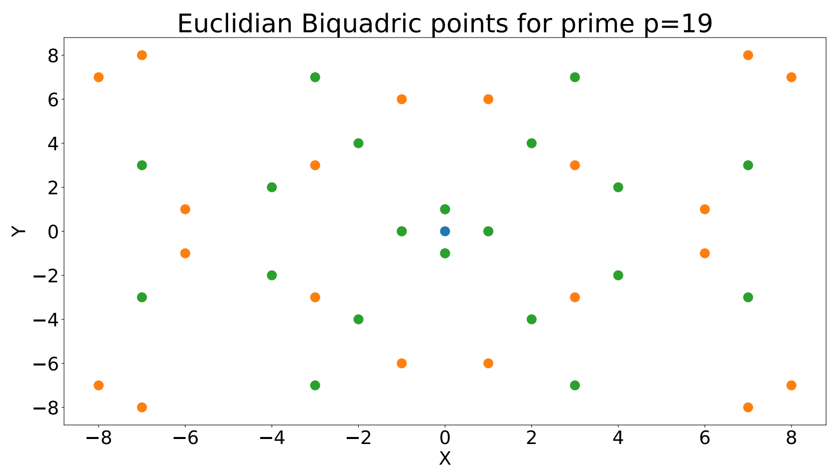


Figure 2: The Minkowski biquadric (top) and the Euclidian biquadric (bottom). The blue point represents the centre of which the rest of the points in green and orange are neighbors. The points shown here satisfy the quadric equations for the matrix forms given in (2.25) and (2.26).

2.4 Form and Types of Transformations

The goal of this thesis is to look at the transformations of the space, in particular of the so called Lorentz and the Gauge transformations. For this we now define the action of projectivities on points, hyperplanes, and quadrics.

Definition 2.33. *Transformation of points*

*Let Π be a projectivity represented by the matrix ρ. Then for any point p in the point set **P**, we have:*

$$p' = \rho p \implies p'^i = \rho^i_j p^j, \quad \forall i, j \in \{0, \dots, n\} \tag{2.28}$$

Theorem 2.7. *Transformation of hyperplanes*

For a given projectivity Π acting on the projective space PG(n,q), the action of the projectivity on any hyperplane H is given by:

$$H' = \rho^{-T} H \tag{2.29}$$

Proof. *The proof of above uses the fact that projectivities keep the incidence structure invariant. Therefore, Let point $p \in H$, and the new point p' be in H' then:*

$$\begin{aligned} & p'^T H' = 0 \\ \implies & (\rho p)^T H' = 0 \\ \implies & \rho^T H' = H, \\ \implies & H' = \rho^{-T} H \end{aligned}$$

$\square$

Corollary 2.5. *The transformation of the representation matrix of quadric Q follows similarly as $Q' = \rho^{-T} Q \rho^{-1}$*

Types of Transformations

Since we have discussed how points, their duals, and quadrics transform, and we have the spaces within which we wish to work, we can also talk about what kind of transformations might exist. In particular the discussion about *projectivities* has already introduced the concept of *automorphisms* of the space. The representation of a projectivity can be represented by the matrix form:

$$\rho = \begin{pmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{h}^T & \kappa \end{pmatrix} \tag{2.30}$$

Due to projective nature of the space and the breakdown into affine subspaces, we can set the element κ such that the projective groups are the factor groups of the generalized groups over vector spaces, with the factor being the projective identity group, such that a transformation ρ is the same as another up to a scale factor $\kappa \in F_p$.[15]

The *projectivities* can then be broken down further, and we define the further groups of Affinities.

[15]This is usually set as 1.

Definition 2.34. *Affinities*

Affinities $\mathscr{A} \subset \Pi$ which define the group of automorphisms of affine subspaces in the projective space. They can be represented by the matrix from:

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{pmatrix} \tag{2.31}$$

such that we have for a point in the affine space $P = (p, 1)^T$: $P \to \boldsymbol{A}p + \boldsymbol{t}$.

In particular the *Affinities* have two subgroups of *affine transformations* and *translations*.

Definition 2.35. *Affine Transformations*

The affine transformations are the subgroup of affinities without the translation vector, such that we have the form:

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0}^T & 1 \end{pmatrix} \tag{2.32}$$

Definition 2.36. *Translations $\boldsymbol{T}$*

The subgroup of Translations are defined by the matrix form:

$$\begin{pmatrix} \boldsymbol{I} & \boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{pmatrix} \tag{2.33}$$

where we have the translation vector $\boldsymbol{t}$ such that the points in the subspace are transformed by an addition of this translation vector.

Corollary 2.6. *The group of translations in 1-dimension is isomorphic to the group of integers modulo p, $\mathbb{Z} \pmod{p}$ and are therefore generated by the canonical translations. In 1-dimensions this is the translation with the vector $\boldsymbol{t} = \{1\}$. In 2-dimensions we have a breadown of the group into two 1-dimensional translations which are the canonical translations defined by:*

$$\boldsymbol{t}_t = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{t}_s = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.34}$$

Here we have use subscripts 't' and 's' to represent translations along the time and along the space axis respectively. Higher dimensional counterparts follow similar canonical generators.

Remark. *The group of affine transformations keep the centre* $\{0, 0, \ldots, 1\}^T$ *invariant and include Boosts and rotations. On the other hand translations do not keep the centre invariant, and the entire affine subspace can be reached by these transformations.*

Finally another important transformation is represented by the *tilt* vector, which act on the hyperplane at infinity. These are a subgroup of the projectivities.

Definition 2.37. *Tilts* $\boldsymbol{H}$

The subgroup of tilts are defined by the matrix form:

$$\begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{h}^T & 1 \end{pmatrix} \tag{2.35}$$

where we have the tilt vector $\boldsymbol{h}$*. The tilt vector is defined by it's action on the hyperplane at infinity, such that for the hyperplane* $H^\infty = \{\boldsymbol{0}, 1\}^T$*, we have the new hyperplane given by* $\{\boldsymbol{h}, 1\}^T$*.*

3 The case for Finite Projective Geomeries: Motivations through a symmetric lens

> **But in my opinion, all things in nature occur mathematically.**
>
> *René Descartes*

Before moving on to talk about the symmetry group representations which exist in our spacetime, it is fundamental to understand the setting and to ask the following questions.

Why might finite projective geometries be a good way of understanding nature? In particular, what roles do the terms *finite* and *projective* have to play in this understanding?

These questions and certain motivations were already glimpsed at in the introduction. In section 2 we discussed what finite and projective means, both synthetically (set-theoretic) and mathematically (coordinates). But we have not yet understood completely what it is we are looking for. What is a lorentz transformation and why is it important? What is a gauge transformation, and why is it important? The import of these ideas from the theories of gravitation and quantum mechanics makes sense. But there is a need to understand how the continuous symmetries correspond to the case of finite space. In this section, we will discuss these ideas in general, before moving on to a more mathematical treatment in the next sections.

At its simplest then we have a geometry which is a set of points and lines. The points and lines follow an incidence relation, and this tells us which point (line) lies on which line(point). We limit then the number of such points and lines that can exist in our space. Furthermore we define at each point an object called a 'biquadric' which allows us to define a 'neighborhood'. The neighborhood is important because it gives us a way to mathematically represent causality; what comes before, and what comes afte.r[16]The biquadric is then the element in our space, which is analogous to the metric tensor in GR.

The first idea *in favour of* finite geometry is the equivalence of quadrics (pg. 21 of [Mec]).

Theorem 3.1. *All quadrics in even dimensions are equivalent. The choice of using the signature (1,3) is arbitrary.*

Proof. Quadrics are objects which are kernels of second order polynomials. In the case of the Galois Fields, Sylvesters Law of Inertia is extended. This is because one

[16] The idea of what is before and what is after is not yet distinct.

has now solutions for equations of the form:

$$\sum_{i=1}^{n} a_i^2 = -1 \tag{3.1}$$

This allows us to change the signature of any matrix, such that the canonical forms are all equivalent to the Minkowski case.

□

The Minkowski quadric itself is also equivalent to the Euclidian quadric given by the pre-metric in equation (2.27). However, when the transformations required are for biquadrics $Q^{\pm}$, the allowed projectivities are fewer, since now both the quadrics must transform. In this case, the two pairs of biquadrics we are left with are the canonical biquadrics given by equations (2.25) and (2.26).

Example 3.1. *In the Galois Field with p=5, we have:*

$$2^2 = 4 = -1 \pmod 5$$

In case of the normal Lorentz transformations then one has the projective Lorentz transformations, which keep the quadric and the point it is centred on invariant. This is the condition of keeping the neighborhood of a point invariant, and therefore, one of the central symmetry arises from the symmetries of the central object: *The biquadric*.

The second idea which relates especially to the present thesis is the idea of a fiber space emerging out of the geometry itself. We build upon it here.

Theorem 3.2. *There exists a local domain of points and points outside of the local domain can be mapped back in.*

Proof. A Galois field is a periodic field and consists of equivalence classes. This means that for any $n_1, n_2 \in F_p, n_1 + n_2 \longrightarrow (n_1 + n_2) \mod (p) = n_3$. *Such that* $n_1 + n_2 \sim [n_3]$. *Furthermore, let* $n_i \in F_p$, *and let* $n_i > \sqrt{p}$. *Then we have* $S = n_i^2$ *and* $S > p$. *Therefore* $\sqrt{S} = \sqrt{S \mod (p)}$.

This means that for elements greater than $\sqrt{p}$, *the square root of the square will be mapped back to another element which is not neccessarily the same. This is of fundamental importance, since we calculate distance with squares using the quadric (see Lorentz transformations).*

□

Therefore we have now shown the existence of a local world domain existing below certain values of the field ($\sqrt{p}$ in the 1-dimensional case) . This means that points outside the local domain must in some way be mapped back to the points. So for a neighborhood of a point C we have elements at a unit distance both inside and outside the local world domain. However, these points themselves are different.

Definition 3.1. *Fiber Space*

The fiber space of finite projective geometry is the hyperplane connecting 'similar' points outside the Local world domain to the point in the local world domain they correspond to. This is a re-folding of the space such that the local world domain has a geometric connection to the global domain.

Remark. *Let z be the integer closest to $\sqrt{p}$. In the 2-dimensional affine space the fiber space is the union of $(z)^2$ lines. Similarly for the line at infinity one has z lines.*

The existence of a fiber space leads directly to the idea of gauge degrees of freedom, which are in general particle degrees of freedom. In our case we have

Definition 3.2. *The Existence of Gauge Transformations*

For the fiber space, gauge invariance refers to the freedom in choosing how the mapping $GD \longrightarrow LD$ [17]shall exist. Transformations which keep the fiber space invariant are known as the gauge transformations.

One may now ask the question: How does this relate to the gauge symmetries we have so far observed, namely the SU(3)×SU(2)×U(1) symmetry of the standard model. The answer to this question relates to the fact that we work in a projective geometry, but first we theorize about the transformations in the 2-dimensional affine plane.

Definition 3.3. *The gauge transformations in the 2-dimensional affine space are the set of transformations which keep the fiber space of lines, connecting quadric points in the local world domain with quadric points outside it, invariant. Therefore we have the decomposition of a 2-dimensional object into a 1-dimensional one. The group is then denoted $G(1)$.*

Applying the logic above, it is clear that in higher dimensions, the fiber space must consist of planes (3-d) and solids (4-d), such that for each affine space $\mathbb{K}^d$, we have the gauge group $G(d-1)$. The final idea then is of uniting the three groups, which happens naturally in the 4-dimensional case.

[17]GD represents points in the global domain outside of the local domain (LD).

Theorem 3.3. *Gauge transformations of 4-dimensions*

Using equation (2.6) *we can write the 4-dimensional projective space as:*

$$\mathfrak{p}^4 = \mathbb{K}^4 \cup \mathbb{K}^3 \cup \mathbb{K}^2 \cup \mathbb{K}^1 \tag{3.2}$$

Therefore the gauge group can be written as the product: $G(3) \times G(2) \times G(1)$

Now an important question arises: How can we parametrize these gauge transformations? This a fundamental question and the main topic of this thesis relates to this parametrization. The central idea here is to use the *intersection of two quadrics* and to search for those transformations which keep this intersection invariant. But which quadrics and their intersections must we consider? Here we use the concept of a biquadric field. This is the field of biquadrics which is generated across the affine plane by translations of the center point. We choose a center[18]and equip it with a biquadric. The group of translations then generates a biquadric field across the affine plane. [19]We choose one of these translated biquadrics, and in general this can be any translated biquadric. The intersection of these two biquadrics will then be the object of our interest.

Before we move on, it is important to talk about the rest of the reasons why a finite projective geometry is a valid approach to unification. These reasons relate to what we have talked about but will not be expounded upon in the sections to come. Possibly among the most interesting reasons is the **non-existence of ovoids**[20]for dimensions greater than 3 (see pg.24 of [Bro00]).

Theorem 3.4. *The maximal space that can realistically[21]exist is the space with dimension* $d = 4$.

Proof. For projective spaces with dimensions greater than or equal to 4, the higher dimensional light cone can be seen as a cylinder. In these higher dimensions a quadratic shape like a hyperboloid can be seen as a connection of lines, such that it is possible by the use of a coordinate system to suppress the hyperboloid to it's $d < 4$ *dimensional form. The coordinate system one uses is that of the light cone*

[18]In this thesis, the center is assumed to be the point $\{0,0,\ldots,1\}^T$ and the corresponding hyperplane at infinity will be $\{0,0,\ldots,1\}^T$.

[19]These transformations do not generate biquadrics for points at infinity.

[20]An ovoid is the projective equivalent of an oval in real space. In general no 3 points on an ovoid are collinear. An example is the quadric we have discussed before.

[21]We use realistic to distinguish what we see and can live in, as opposed to a mathematical structural existence.

itself, such that points in these higher dimensions are at a distance 0 from each other. Since the light cone has quadric points only at the hyperplane at infinity, the structure of the light cone is one-dimensional less than that of the projective space. This means that the hyperplane at infinity can only have a maximum dimension of 3, and therefore the projective space is only possible for 4-dimensions.

□

Therefore, we have mathematically theorized not only the existence of the 4-dimensional space time, but also the existence of Lorentz Transformations, Fiber space, and the symmetries relating to the fiber space. It is important to stress again that this is possible only in a **finite and projective** space. In the next sections a visualization of Lorentz Transformation will be shown, for a better understanding of how the finite group connects to the continuous one[22]. Afterwards, the intersection of two quadrics shall be expounded upon, with a focus on the study of their symmetries.

[22] In general a finite group can be thought of as being embedded within a continuous one.

4 Lorentz Transformations

> **Nature uses only the longest threads to weave her patterns, so each small piece of her fabric reveals the organization of the entire tapestry.**
>
> *Richard P. Feynman*

The first set of transformations which are defined by the symmetries of the biquadric are the Lorentz Transformations. The name is taken as an analog of the Lorentz group in the continuous spacetime, although a complete one-to-one analogy might not exist. The parametrization of these transformations is done by looking at their specific action on the biquadrics.

Definition 4.1. *Lorentz Transformations $\boldsymbol{L}$*

For a set of quadrics $\boldsymbol{Q}^{\pm}$ centred at point $\boldsymbol{C}$, Lorentz Transformations are the set of those transformations which leave the point set of one of the quadrics and the centre point $\boldsymbol{C}$ of the biquadric invariant.
If $\boldsymbol{P}_Q$ be the point set belonging to the quadric $\boldsymbol{Q}$ with points given by P_Q^i, P_Q^j where $i, j = \{1, \ldots, |\boldsymbol{P}_Q|\}$, then the above condition implies:

$$L_k P_Q^i = P_Q^j, \quad \forall P_Q^i, P_Q^j \in \boldsymbol{P}_Q, \forall L_k \in \boldsymbol{L} \tag{4.1}$$

$$L_k \boldsymbol{C} = \boldsymbol{C}, \forall L_k \in \boldsymbol{L} \tag{4.2}$$

It is clear from the definition of the biquadric then, that the action of the Lorentz Group preserves not only the biquadric structure, but also the line at infinity, which is defined as the polar dual to the centre with respect to the quadric.

As we have discussed in the previous section, this action can be represented using the representation matrices of the quadric (or in general the biquadric).

Definition 4.2. *Lorentz Transformations $\boldsymbol{L}$ on quadric*

For a given quadric representation Q, we define the action of the Lorentz group using their matrix representation as follows:

$$Q = \boldsymbol{L}^{-T} Q \boldsymbol{L}^{-1} \tag{4.3}$$

This can then be further simplified, using the fact that a group has inverses to:

$$Q = \boldsymbol{L}^T Q \boldsymbol{L} \tag{4.4}$$

For the chosen centre $\mathbf{C} = \{\vec{\mathbf{0}}^T, 1\}^T$, we can write the group representation as:

$$\boldsymbol{L} = \begin{pmatrix} L & 0 \\ 0 & 1 \end{pmatrix} \tag{4.5}$$

It is clear that $\mathbf{L} \subset \mathrm{Aut}(Q_{\min})$, specifically those which also keep the centre invariant. More generally, $\mathbf{L} \subset \mathrm{Aut}(Q_{\min})$. This is an important property because it distinguishes Lorentz transformations from other automorphisms, namely from the orthogonal group which keeps the Euclidian quadrics invariant. However, for subspaces of the projective space, an embedding of a Euclidian quadric can exist in a Minkowskian one, and then the group $\mathbf{O}(n-1, q)$ becomes a subgroup of $\mathbf{L}(n, q)$, where n is the dimension of the space. This will be shown for the 3-dimensional case later where spatial rotations given by an orthogonal group are a subgroup of the bigger Lorentz group.

The sub-matrix element L defines the lorentz group, since it keeps the Minkowskian pre-metric given in equation (2.27) invariant. We define this as follows:

Definition 4.3. *Action of L*

For the sub-matrix element L, which defines the Lorentz Transformation, we have the conditon:

$$G_{min} = L^T G_{min} L \tag{4.6}$$

These Transformations can be broken down into different forms depending on the space-time dimension we are working in, as we now see.

4.1 The 2-dimensional Case: Boost

For the two-dimensional case the pre-metric is given by the form $\mathrm{diag}\{-1, 1\}$.

Theorem 4.1. *Representation matrix L for 2-dimensions*

For the 2-dimensional case, the matrix L is given by:

$$L = \begin{pmatrix} \lambda & \sqrt{\lambda^2 - 1} \\ \pm\sqrt{\lambda^2 - 1} & \pm\lambda \end{pmatrix}, \quad \lambda \in F_p \tag{4.7}$$

Proof. We have the equation $L^T G_{\text{min}} L = G_{\text{min}}$, where $G_{\text{min}} = \text{diag}\{-1, 1\}$. For $L = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix}$, we have:

$$-\lambda_1^2 + \lambda_3^2 = -1 \tag{4.8}$$

$$-\lambda_2^2 + \lambda_4^2 = 1 \tag{4.9}$$

$$-\lambda_1 \lambda_2 + \lambda_3 \lambda_4 = 0 \tag{4.10}$$

The parameterization is then that given above. □

These are $|\mathbf{L}| = p$ transformations and are the analogous to the boosts in classical relativity. For the case when Q_{min} is replaced by Q_{euc}, the $\mathbf{L}(n) \to \mathbf{O}(n)$,[23]and we then have the case of rotations. This will become clearer in the next section, but first we look at the action of this group elements. This is given in fig. 3, where we consider the case of successive boosts using one element of the boosts (for which a square root exists). The important observation here is the symmetric nature of both the plots with respect to the x axis, such that half the boosted points are reflections of points along the x axis.

4.2 The 3-dimensional Case: Boosts and Rotations

The three dimensional case is slightly more complex but can be broken down to solutions that apply to 2-dimensional sub-spaces, and therefore can be seen as rotations or boosts, depending on whether the time axis is kept invariant or not.

4.2.1 Rotation

Let the representational form L be written as:

$$L = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \tag{4.11}$$

Then R solves the equation:

$$R^T R = I_{2d} \tag{4.12}$$

[23]In general we always concern ourselves with the Minkowski quadric, however subspaces in the projective space can have induced Euclidian quadrics, and so we note the difference in the two $\text{Aut}(Q^{\pm})$ groups.

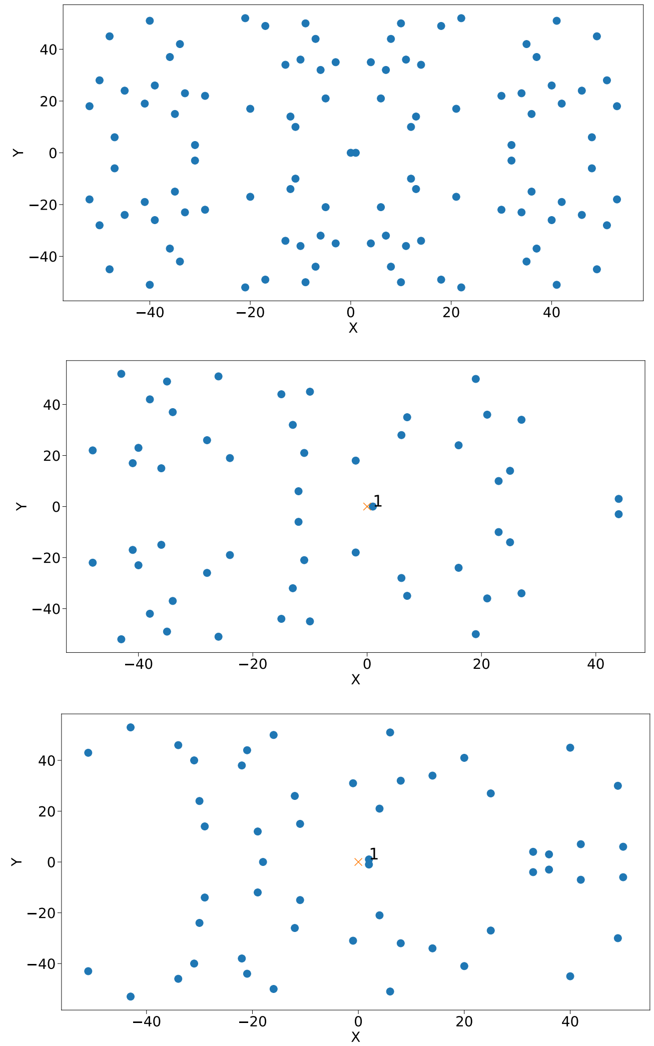


Figure 3: Successive boosts are defined by multiple applications of 1 element of the matrix given in equation (4.7) to point 1 in figure. The action is repeated till point 1 is reached back again. **Top**: Minkowski quadric points **Middle**: Successive boost for quadric point 1. **Bottom**: Successive boost for non-quadric point 1. Seen here is the axial symmetric nature of the boosts for both quadric and non-quadric points.

Therefore R belongs to the group of 2-d orthogonal matrices, and can be represented as:

$$R_1 = \begin{pmatrix} \alpha & \mp\sqrt{1-\alpha^2} \\ \pm\sqrt{1-\alpha^2} & \alpha \end{pmatrix} \tag{4.13}$$

$$R_2 = \begin{pmatrix} \alpha & \pm\sqrt{1-\alpha^2} \\ \pm\sqrt{1-\alpha^2} & -\alpha \end{pmatrix} \tag{4.14}$$

Due to the fact that α can be written as $\cos\theta$ in the continuum limit when the prime $p \to \infty$, this particular set of solutions corresponds to 2-d rotations in a plane. This can be seen even more clearly if we look at the action of Lorentz group on a point $P = \{p_1, P_s, 1\}^T$. We then have:

$$L_i P = \{p_1, RP_s, 1\}^T, \forall L_i \in L \tag{4.15}$$

Here we see that only the spatial part of the point is affected by the Lorentz transformation, and therefore we call them rotations. However, the finite group is only embedded in the bigger continuous counterpart.

An example in the prime field with $p = 43$ can be seen in figure 4 . The quadric points in the 2-d affine sub-space and in the 3-d affine space are shown here. Both cases show the initial quadric as well as the 180 degree rotated quadric, for a sub-set of the quadric points (second coordinate positive). For the entire quadric point space, any rotation gives back the same quadric, and therefore only a sub-set of points is shown here.

The invariance can be further seen for a counter-clockwise rotation of 90 degrees. For the points on the positive second axis, this corresponds to an overlap with the original quadric points. This can be seen below in figure 5, where we plot those points for which the time coordinate is the same. Of note here is the fact that the quadric points shown here belong to the 2-dimensional Euclidian quadric, and hence the difference with the case when one might look at a slicing with the Minkowskian quadric (as seen in figure 3). These are the 2 types of embeddings in the subspaces which were mentioned before.

Finally, to complete the discussion, we can talk about the rotation parameter α. While $\alpha \in F_p$ can always be chosen amongst the p elements, a corresponding square root doesn't necessarily always exist. One can however still find roots which then give us different rotation matrices, such that a full rotation is completed in steps. For instance, in the prime field $\mathbb{F}_4 3$ (3 (mod ()4) = 1, so -1 is a non-square), one has 22 elements having roots. One can also show that these roots can be divided into two sub-sets where

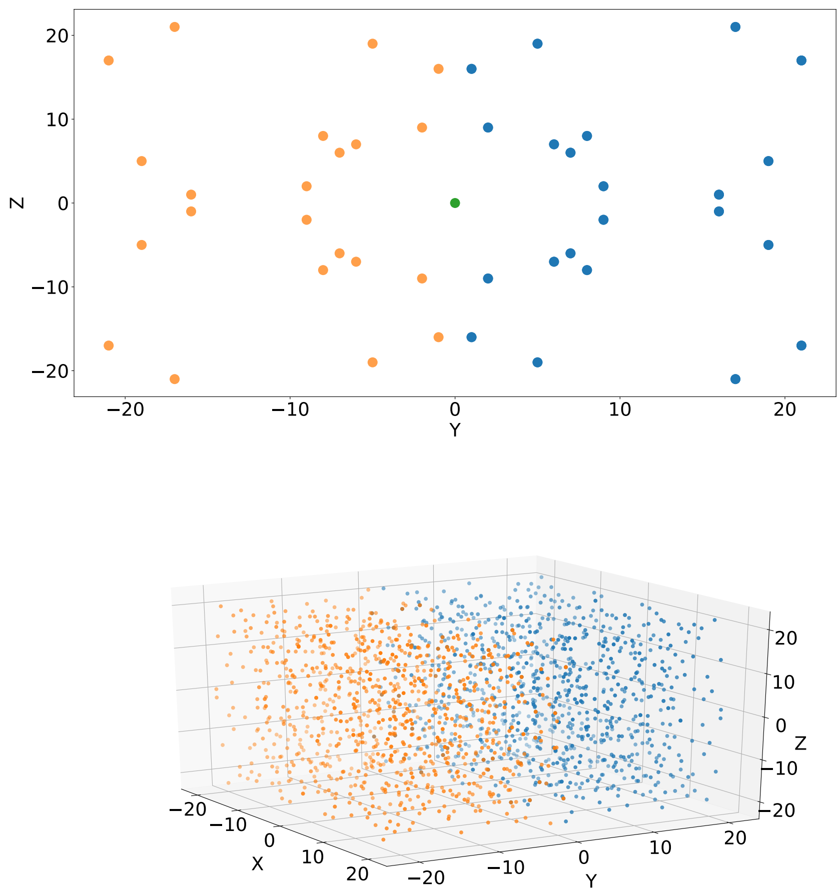


Figure 4: The Minkowski quadric and the corresponding 180 degree rotated quadric for a sub-set of points in 2-d (slice of 3-d) and 3-d spaces. The rotation is an element of L given by equation (4.11), and the action is defined in equation (4.15). Results are for prime p=43.

the sub-set has positive and negative roots. The entire sets of matrices forms the complete group of rotations. In the figure 6, one can see the application of these rotation matrices on a point of the quadric (point labeled as 1). In the 2-dimensional projection of the three-dimensional case (taking a slice of the 3-d points), one can compare and see that all the quadric points that lie in a plane can be generated by all possible rotations of a quadric point.

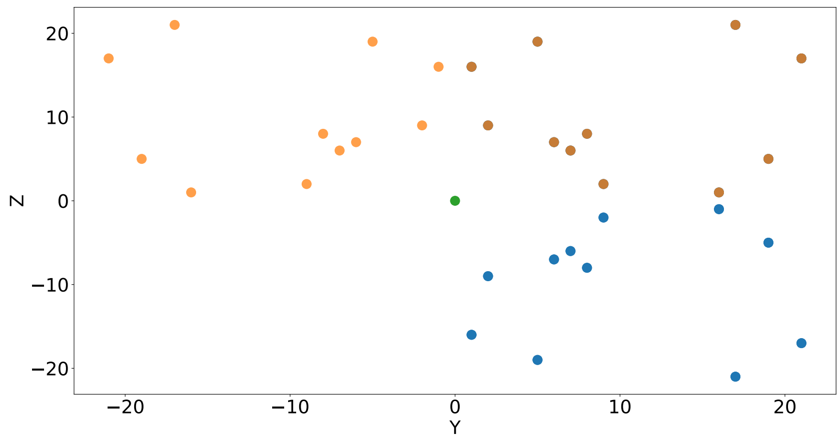


Figure 5: The Minkowski quadric and the 90 degree rotated quadric for a sub-set of points in 2-d showing the overlap of quadric points. Results are for prime p=43.

4.2.2 Successive Rotations and the Local World Domain

Another interesting case of rotation is the case of succesive similar rotations, which in the representational space means the consecutive application of the same rotational matrix to a point. Suppose we have any element L_i of the Lorentz Group such that the form of L_i is given by equation (4.11), where we have the representational matrix R_i. The successive action of such a rotation can be described as,

$$L_i^m = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0}^T & R_i^m \end{pmatrix} \tag{4.16}$$

where we have for the m^{th} rotation the form above.

Figure 7 shows the symmetric quality of these succesive rotations for an arbitrary value of the rotational parameter α.

An important feature of the finite fields lies in the ability to model real space-time in terms of what is called the local world domain. This was studied by both [Hö18] and [Gim18]. The local world domain is defined in terms of coordinate values which lie between $\mp\sqrt{p}$, where p is prime. The idea of having such a domain comes from the fact that due to the finiteness of the field, the order relation which exists due to biquadric points can be destroyed

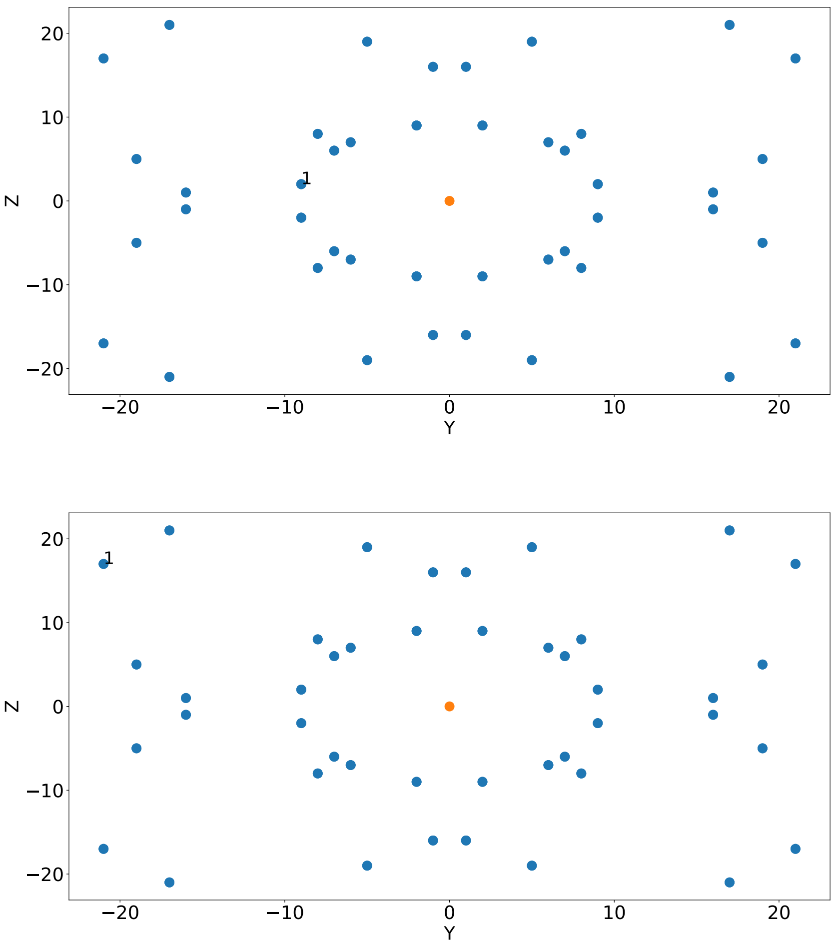


Figure 6: The application of all elements of the Lorentz rotations are shown here for prime p=43. We apply every possible element for which a square root exists in equations (4.13) and (4.14). **Above:** All points generated by the complete set of rotations on point 1. **Below:** The 2-dimensional projection of the quadric points.

in cases where we want to work with second-order quantities. An example of such a quantity is distance, and while distances in finite projective geometries often use cross-ratios, one can also define a quadric distance. Without a lack of generality, we can consider the distance of a point P from the centre

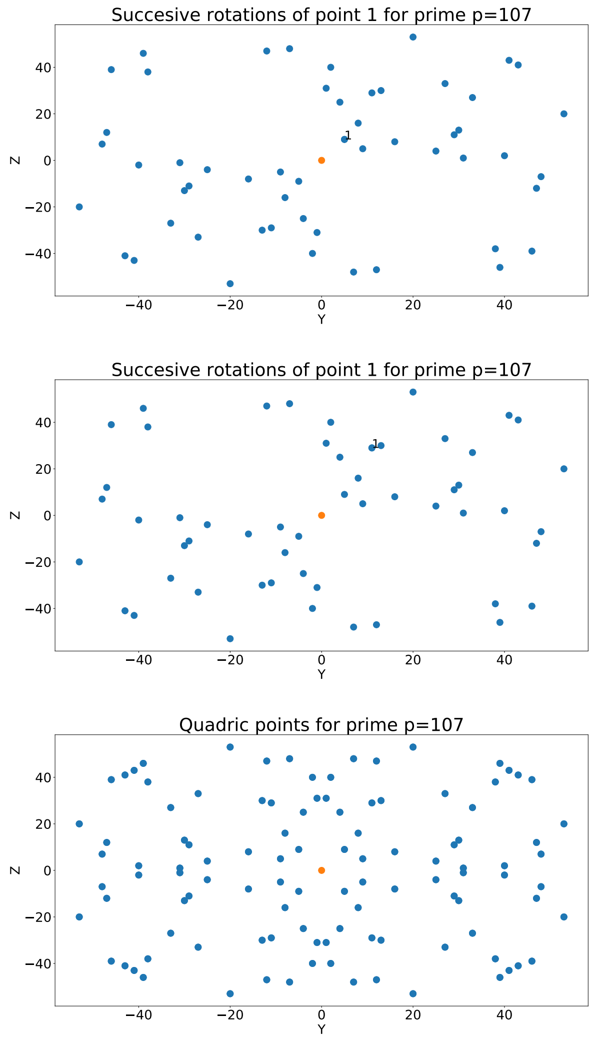


Figure 7: Points generated by application of consecutive rotations as given by (4.16). **Top:** Rotations of point 1 for both y, z coordinates less than 10. **Middle:** Rotations of point 1 for both y, z coordinates more than 10. **Bottom:** Projective quadric points for prime 107.

of our geometry by defining it as:

$$\mathbf{D}^2 = |(G_{ij} p^i p^j) \pmod{p}| \tag{4.17}$$

where the G_{ij} are elements of the Minkowskian pre-metric given by equation (2.27), and p^i and p^j are the coordinates of the point P. It is clear then that for the quadric Q^{+}_{min} at the centre any affine quadric point has a unit distance to the centre.

One can then see that for the two-dimensional sub-space, in which the time coordinate is kept constant, the above distance is essentially just the cartesian-distance of coordinates. In the real space-time then, the group of rotations keeps this distance invariant. Therefore, if one were to choose a point in the 2-d space, the application of consecutive rotations would lead to a circle in this space. However, as is clear from figure 7, it doesn't seem like this is the case with finite fields. This is indeed true to a certain extent since one can see that the values of the points are not lying in a finite circle.

But, the distance measure of these points is the same. Points with coordinate values greater than $\mp\sqrt{p}$ will have their distances mapped back to lower values due to the modulo operation, since any value greater than the square root of the prime will be greater than the prime, when squared, and therefore the modulo operation shall bring the cartesian distance back to it's lower values. A good case study for this is for values of bigger primes. One can then see what happens in the domain defined with coordinate values less than the square root of the prime. Figure 8 shows the local world domain for a prime p=10007. There exist 4 points in this domain, all of whom are at the same cartesian distance from the centre.

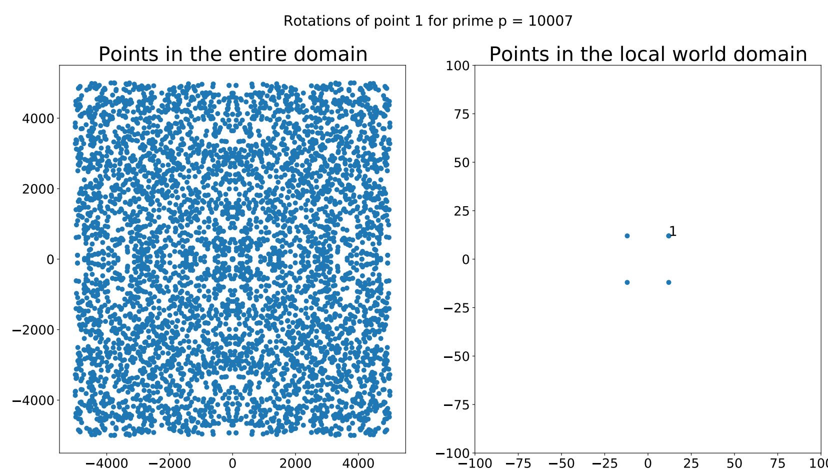


Figure 8: Rotations showing points in the complete space, and in the local world domain

4.2.3 Boost

To get the complete Lorentz group in 3-d one must also have the second case which keeps one of the space dimension invariant. This representation can be written as:

$$L = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \tag{2.9}$$

In this case B solves the same equation as the 2-d Lorentz case and is given by L_{2d} as in equation (4.7). These are also called boosts, since the form for B can also be written in terms of $\sinh(\theta)$ and $\cosh(\theta)$, if λ is instead parameterized as $\cosh(\theta)$.
Therefore, the 3-dimensional case breaks down into the case of 2-dimensional subspaces, such that we have embeddings of the 2-dimensional Minkowski and 2-dimensional Euclidian biquadrics. The Lorentz transformation in 3-dimensions are then the union of those transformations which keep these embedded subspaces invariant. In particular we have two boosts (since we have 1 temporal and 2 spatial directions) and 1 rotation (for the two spatial directions). Therefore the 3-dimensional Lorentz transformations is a 3 parameter group.

4.3 Extension to 4-dimensions

Our discussion above allows us to extend the results for the 4-dimensions. The Lorentz group in 4-dimensions can again be broken down into 3 boosts (1 temporal and 3 spatial directions), and 3 rotations (between the 3 spatial directions), such that we have a 6 parameter Lorentz group. Since the discussion for both boosts and rotations has already taken place, the 4-dimensional case will not be probed further. We move therefore to the discussions about gauge transformations, and the structures underlying them.

5 Gauge Transformations

If I have seen further, it is by standing upon the shoulders of giants

Sir Isaac Newton

An even more interesting aspect of symmetries are related to the gauge transformations. In this work we will look at transformations which keep the intersection of two quadrics invariant. Here the initial discussion of translation becomes important, as we will try to preserve the structure for the case when the second quadric is the translated first quadric.

Translations and Intersections

Theorem 5.1. *Translated quadric*

For a quadric representation Q with a pre-metric G centred at $\boldsymbol{C}$, the translated quadric is given as:

$$Q'_{trans} = \begin{pmatrix} G & -G\boldsymbol{t} \\ (-G\boldsymbol{t})^T & 1 + \boldsymbol{t}^T G\boldsymbol{t} \end{pmatrix} \tag{5.1}$$

and the new centre is given as:

$$\boldsymbol{C}_{trans} = \{\boldsymbol{t}, 1\}^T \tag{5.2}$$

Proof. For the case of translations, the quadric can be written as $Q_{trans} = \mathbf{T}^{-T} Q \mathbf{T}^{-1}$. Then since $\mathbf{T} \subset UT(n+1)$[24], the inverse is given by the mapping: $\mathbf{t} \to$ **-t**. The transpose is then the subgroup of $LT(n+1)$[25], and the matrix multiplaction gives the expression above.
Similarly since $\mathbf{T} \subset \mathscr{A}$ the group of affinities given by equation (2.31), the last coordinate of the centre remains invariant, and the affine coordinates are given by the translation vector. □

In general for translations along a line, that is the case of successive translations, one can define the 'direction' of translations by the hyperplane joining the two centres.

Definition 5.1. *The hyperplane of centres*

[24] Group of Upper Trinagular matrices of $((n+1) \times (n+1))$ dimensions
[25] Group of Lower Triangular matrices

The hyperplane of centres is the hyperplane connecting the two centres. It is defined as the cross-product of the two centres and we denote it by $\boldsymbol{H}_c$. Then:

$$\boldsymbol{H}_c = \boldsymbol{C} \times \boldsymbol{C}_{trans} \tag{5.3}$$

we calculate the cross-product for the two given centres using equation (2.13), *to get:*

$$\boldsymbol{H}_c = \begin{pmatrix} -t_2 \\ t_1 \\ 0 \end{pmatrix} \tag{5.4}$$

Remark. *The hyperplane of centres is named as such since it defines all the possible future and past centres of our translated quadrics.*

For the quadric points which belong to both the quadrics such that $P \in Q_{\text{min}} \cap Q_{\text{trans}}$ then solve simultaneously:

$$P^T Q_{\text{min}} P = 0, \quad P^T Q_{\text{trans}} P = 0 \tag{5.5}$$

5.1 Symmetries in the 1-Dimensional Projective Space

The first case we have is of the 1-dimensional case, wherein the quadric is a 2×2 matrix, and we have 2 points in the affine plane, and the point at infinity is also in the quadric. A translation of the quadric can intersect in 1 point at the affine plane, such that the intersection has 2 total points.

Definition 5.2. *Intersection of 1-dimensional quadrics*

For the quadric given by diag$(-1, 1)$ *and the translation by the generator* $\boldsymbol{t} = t$, *we have the intersection as the solution of the equation:*

$$p_2(tp_2 - 2p_1) = 0 \tag{5.6}$$

Thie solution is for any general point with coordinates $\{p_1, p_2\}^T$ *and only has a solution when t=2. Then we have the point in the affine plane* $(1, 1)$ *with* $(p-1)$ *scalings in the 2-d vector space representing* $\mathfrak{P}K^1$, *and the point at infinity* $P^\infty = \{1, 0\}$.

The group of transformations keeping both the points (intersection point and the point at infinity) invariant is just the identity group. The transformations keeping only the affine point invariant are $p-1$ permutations of the other $p-1$ affine points isomorphic to the group $\mathbb{Z}^*/\mod(p)$, given as:

$$\begin{pmatrix} g & 1-g \\ 0 & 1 \end{pmatrix} \tag{5.7}$$

However as we have seen, only one particular translation (and it's inverse) have an intersection of two quadrics. In the general case, there is no such intersection, except for the point at infinity. This is why we look first at the transformations keeping the quadric invariant.

As we have seen before, in the 2-dimensional and higher we have the Lorentz group. However, the Lorentz group is the group of automorphisms of the biquadric. For only one quadric, we have the automorphism group $\mathrm{Aut}(Q^+)$, and in particular this has a subgroup of transformations which keep the quadric invariant, but not the centre.

For the 1-dimensional case we have the group[26]:

$$G(1\mathrm{D}): \begin{pmatrix} \lambda & \sqrt{\lambda^2-1} \\ \sqrt{\lambda^2-1} & \lambda \end{pmatrix} = \begin{pmatrix} 1 & \lambda \\ \lambda & 1 \end{pmatrix}, \quad \sqrt{\lambda^2-1}/\lambda \to \lambda \tag{5.8}$$

where we have redefined the parameter to get the projective equivalent. We have then that $G(1\mathrm{D}) \subset \mathrm{Aut}(Q^+)$ with the effect $G\mathbf{C} = \{\lambda, 1\}^T$. This group shall later serve as an inspiration for higher dimensional equivalents.

5.2 Symmetries in the 2-Dimensional Projective Space

We look at the general case of the 2-dimensional translated quadric and it's intersection with the initial one. Then writing the translation vector $\boldsymbol{t}$ as $\{t_1, t_2\}^T$, the translated quadric is given as:

$$Q_{trans} = \begin{pmatrix} -1 & 0 & t_1 \\ 0 & 1 & -t_2 \\ t_1 & -t_2 & 1+t_m \end{pmatrix}, \quad t_m = t_2^2 - t_1^2 \tag{5.9}$$

According to (5.5) using homogeneous coordinates $P = (p_1, p_2, p_3)^T$ for the quadric points we now have:

$$-p_1^2 + p_2^2 + p_3^2 = 0, \tag{5.10}$$

$$-p_1^2 + p_2^2 + (1+t_m)p_3^2 + 2t_1p_1p_3 - 2t_2p_2p_3 = 0 \tag{5.11}$$

The solutions of this equation and the spaces they are part of have multiple elements. We divide the discussion now into 4 parts: *Points at Infinity*, *Points in the Affine Plane*, *The Hyperplane of Intersection*, and the *Interunion*.

[26]These are not all the elements, since we have also the case when the last matrix element is the negative of the first. Here we focus only on this subset, and the results carry over to the other elements.

Points at Infinity

As was mentioned before, every quadric has two points at infinity. Since $\mathbf{T} \subset \mathbb{A}$, these points are unmoved due to translations. We have the two points $P_{\text{inf}} = \{\pm 1, 1, 0\}^T$. All the other points at infinity which solve equation in homogeneous coordinates are equivalent to these two points. Here these points correspond to the hyperplane of intersection which is the line at infinity (upto a factor λ):

$$H_{\text{inf}} = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{5.12}$$

In general the *affinities* also belong to the group of automorphisms of the hyperplane at infinity $\text{Aut}(\mathbf{H}^\infty)$. In particular there exist the 'translations at infinity', which are those transformations which act on the affine subsection of the hyperplane at infinity.

Points in the affine plane

Apart from the two points at infinity one has two points in the affine plane, given by,

$$P_{\text{aff}} = \{P, 1\}^T \tag{5.13}$$

The points set of these 4 intersection points is denoted $\mathbb{P}_{\text{ins}}$, and $\mathbb{P}_{\text{ins}} = P_{\text{aff}} \cup P_{\text{inf}}$.

The group $\text{Aut}(\mathbb{P}_{\text{ins}})$ consists only of the identity element.

Hyperplane of Intersection

We consider now the points solving the equations of intersection, and in particular for those solutions which have $p_3! = 0$. Subtracting the two equations given by (5.11) we have:

$$2p_1 t_1 - 2p_2 t_2 + p_3(t_m) = 0 \tag{5.14}$$

This equation defines the hyperplane spanned by the two affine points of intersection. We call this plane the hyperplane of intersection $\mathbb{H}_{\text{ins}}$.

Definition 5.3. *Hyperplane of Intersection*

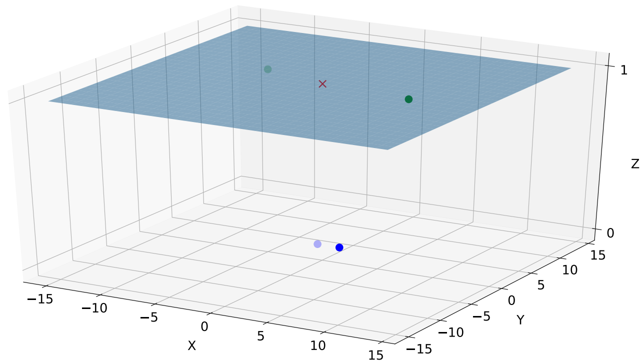


Figure 9: The 4 points of intersection. The two green points represent the affine points (given by the shaded plane), and the two blue points are the points at infinity (for $t_1 = 2$ and $t_2 = 1$) and prime $\mathbf{p} = 31$. The cross is the centre.

For the difference of two quadratic equations the points $P_{aff} = \{P_{aff}, 1\}^T$ span a hyperplane given by:

$$\mathscr{H} = \begin{pmatrix} 2t_1 \\ -2t_2 \\ t_m \end{pmatrix} \tag{5.15}$$

The hyperplane has $p+1$ total points, with p points in the affine plane and 1 point at infinity.

Lemma 5.1. *$\mathscr{H}$ intersects the quadrics in the affine plane (given in inhomogenous coordinates) at the two points defined in the previous section.*

Lemma 5.2. *For the 2-dimensional case with the hyperplane of intersection, one can also solve the cross-product of two points which satisfy* (5.14)*. In the affine plane with p_3 set as 1, we choose the two points:*

$$P_1 = \begin{pmatrix} 0 \\ \frac{t_2^2 - t_1^2}{2t_2} \\ 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} \frac{-(t_2^2 - t_1^2)}{2t_1} \\ 0 \\ 1 \end{pmatrix} \tag{5.16}$$

The hyperplane of intersection is then given by equation (2.15).

$$\mathscr{H} = det\begin{pmatrix} 0 & \frac{t_2^2-t_1^2}{2t_2} & 1 \\ \frac{-(t_2^2-t_1^2)}{2t_1} & 0 & 1 \\ \hat{e_1} & \hat{e_2} & \hat{e_3} \end{pmatrix} \tag{5.17}$$

We then get the same representation as in (5.29).

In the *homogenous coordinates* the hyperplane of intersection is a plane and we plot it in figure 10 below.

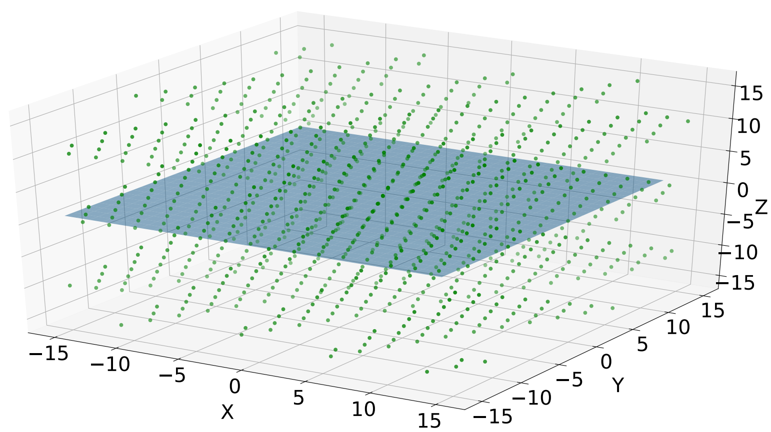


Figure 10: The hyperplane of intersection as given by equation (5.29) in the homogenous coordinates for the prime p=31 and translation vector $\mathbf{t} = \{2,1\}^T$. The affine plane is the shaded region.

The plane itself shows the periodicity of the galois field, since in the 3-dimensional vector space the plane looks like a series of planes. This also defines the line in the affine plane which is where the scaled out coordinates of the plane appear, as can be seen in the figure 11 below.

Before we move on to the next topic, it is important to discuss also the second quadric which together makes up the biquadric. The existence of the hyperplane of intersection is not limited just to the quadric we have so far discussed. The second quadric also has intersection point with its translated counterpart. For the quadric Q^- the translated quadric can be calculated

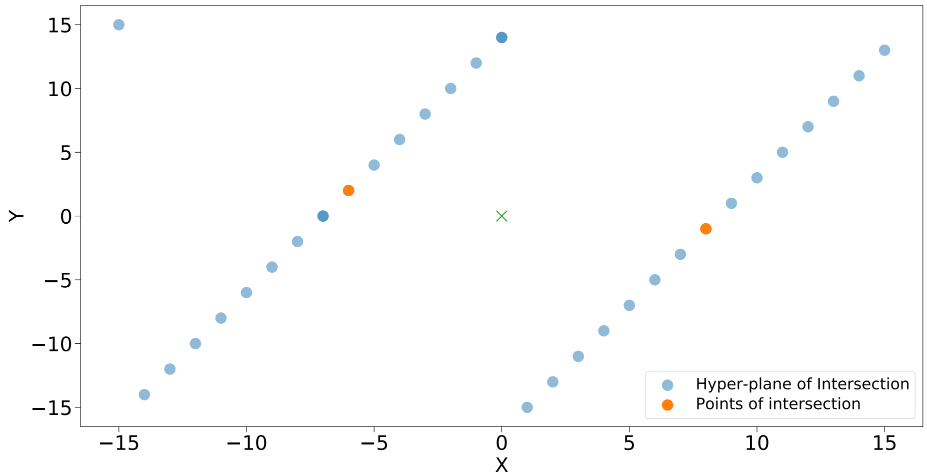


Figure 11: The hyperplane of intersection as defined in equation(5.29) for the translation vector $\mathbf{t} = \{2, 1\}^T$ and p=31. Here the points in the affine plane are shown, along with the two affine points which span the line, given in orange.

using equation (5.1) by replacing the 1 with a non-square *ns* and it is given as:

$$Q^{-} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & ns \end{pmatrix} \implies Q^{-}_{\text{trans}} = \begin{pmatrix} -1 & 0 & t_1 \\ 0 & 1 & -t_2 \\ t_1 & -t_2 & ns + t_m \end{pmatrix} \tag{5.18}$$

We see then, that the intersection points of the second quadric also lie on $\mathscr{H}$.

Union of intersection planes

A more general structure exists for the equations. We call this the *Union of Intersection Planes*, and it is the structure which includes also the hyperplane at infinity along with the hyperplane of intersection. The generalized form represents the difference of two quadratic equations

Definition 5.4. *Union of intersection planes*

For the difference of two quadratic equations defining the quadrics the point set $\mathbb{P}_U$ defines the Union of Intersection planes (Interunion from here on). It is given as the union of the hyperplanes, and defined by the equation:

$$p_3(2p_1t_1 - 2p_2t_2 + p_3(t_m)) = 0 \tag{5.19}$$

The explicit form can be written as a matrix as:

$$\mathscr{I}_U = \begin{pmatrix} 0 & 0 & t_1 \\ 0 & 0 & -t_2 \\ t_1 & -t_2 & t_m \end{pmatrix} = \begin{pmatrix} \boldsymbol{0}_{2\times 2} & -G_{min}\boldsymbol{t} \\ -(G_{min}\boldsymbol{t})^T & t_m \end{pmatrix} \tag{5.20}$$

In general the representation on the right can be extended to greater dimensions.

Corollary 5.1. $\forall P_U \in \mathbb{P}_U$ *we can write equation* (5.19) *as:* $P_U^T \mathscr{I}_U P_U = 0$.

Corollary 5.2. *The order of the Interunion* $|\mathscr{I}_U| = p + p + 1 = 2p + 1$.

Remark. *In general for any affinity, the span of the two points at infinity will always be the line at infinity. This means that the interunion will always contain the line at infinity, while the hyperplane of intersection will change depending on the transformation.*

The union is plotted below, where we add the line at infinity to the hyperplane of intersection.

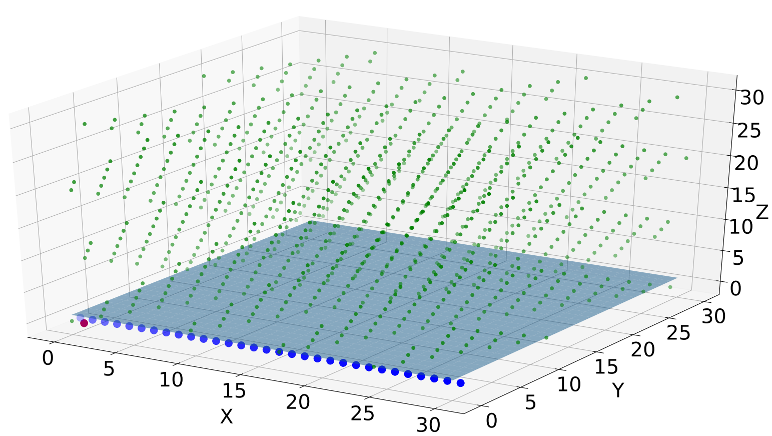


Figure 12: The union of intersection planes in the homogenous coordinates for the prime $p = 31$ as given by equation (5.19) for the translation vector $\mathbf{t} = \{2, 1\}^T$. The blue points represent the points at infinity with the red point being the point at infinity given as $P^\infty = \{1, 0, 0\}^T$.

5.2.1 Symmetries of the hyperplane of intersection

A centre with a quadric defined on it allows us to define a neighborhood. The Lorentz group has been discussed as being the $\mathrm{Aut}(Q, C)$ which for the case of unique centre is also the $\mathrm{Aut}(Q^{\pm})$.[27] For the consideration of gauge invariances and gauge degrees of freedom, the idea of the intersection of two quadrics comes into play. In general, for the affine plane, especially in the flat field approximation[28], one can use the translations to generate a flat biquadric field. Since we already have the group of automorphisms of the quadrics, we know first concern ourselves with the automorphisms of the hyperplanes (and later of the interunion). This is the group $\mathrm{Aut}(\mathscr{H})$ which keep the hyperplane of intersection invariant. In particular we will be interested in those elements of the group which have some other symmetries associated with them as well.

A general translation is given by the translation vector $\mathbf{t}$, however as was discussed before, this group is generated by the two elements $\mathbf{t}_t, \mathbf{t}_s$ as given by equation (2.34). Therefore we will first simplify the results and discussions by looking at the case for one of these generators. A more general outlook for an arbitrary translation will be provided later. To show however that the discussion holds for any translation we provide the following argument.

Theorem 5.2. *Extension of Generators*

The set of transformations which define how the gauge transformations themselves transform for every point in the affine plane are a sub-set of the projectivites without the translations. Formally, the set of matrices $\boldsymbol{R}$ *which allow* $\mathscr{H}(t_1, t_2) = \boldsymbol{R}^{-T}\mathscr{H}(1,0)\ \forall t_1, t_2 \in F_p$ *have the form* $\boldsymbol{R} = \begin{pmatrix} R & \boldsymbol{0} \\ \boldsymbol{0}^T & 1 \end{pmatrix}, R \in M\{2\times 2, F_p\}$.

Proof. Here we use the fact that we have so far defined $\mathscr{H}$ as the intersection plane spanned by the points of intersection of the quadrics centred on $\mathbf{C}$ and another $\mathbf{C}_{\text{trans}}$. It is sufficient for the proof then that the group of transformations we are interested in keeps the centre invariant, and maps $\mathbf{C}_{\text{trans}}$ to a different point, which is defined for a different translation. Such that we have for a set of transformations $\mathbf{R}$:

$$\mathbf{RC} = \mathbf{C} \tag{5.21}$$

$$\mathbf{RT}_1\mathbf{C} = \mathbf{T}_2\mathbf{C}, \forall \mathbf{T}_1, \mathbf{T}_2 \in \mathbf{T} \tag{5.22}$$

[27] It is possible in our geometry to have biquadrics which have multiple centres, such that the different centres have the same neighborhood. See for instance section 3 in [Las14].

[28] All the points in the affine plane are in the same number of neighborhood. This can be generated using translations for the affine plane.

The set of transformations following this are given by:

$$\mathbf{R} = \begin{pmatrix} R & \mathbf{0} \\ \mathbf{0}^T & 1 \end{pmatrix}, R \in M\{2 \times 2, F_p\} \tag{5.23}$$

$\square$

Remark. *These transformations allow us to find the hyperplanes of intersection between the origin and the required point. For a more general case of shifting between two arbitrary points, the transformations will be the set of all projectvities.*

In the following discussion then we restrict ourselves to the reduced representation, where we choose $\mathbf{t}_t = \{1, 0\}^T$ as our translation vector. For this case, the hyperplane of intersection is given in figure 13.

The automorphisms can then be represented and broken down into two types, which we call type 1 and type 2, which are distinguished according to the scaling of the matrix representation. A further distinction appears later in our discussion as well. These transformations are given as:

$$\mathbf{G} = \mathrm{Aut}(\mathscr{H}) \subset M_{(n+1)\times(n+1)} \tag{5.24}$$

$$\mathbf{G} = \mathbf{G}_1 \cup \mathbf{G}_2, \text{where,} \tag{5.25}$$

$$\mathbf{G}_1 \in \begin{pmatrix} \Lambda & \alpha \\ \beta^T & 1 \end{pmatrix}, \mathbf{G}_2 \in \begin{pmatrix} 1 & \alpha^T \\ \beta & \Lambda \end{pmatrix} \tag{5.26}$$

Transformations of Type 1

The first transformations are those we call type 1 transformations which involve the last element of the matrix to be set to 1. We represent these with a '1' in the subscript. Then the group $\mathbf{G}_1$ solves

$$\mathbf{G}_1^{-T} \mathscr{H} = \mathscr{H} \tag{5.27}$$

and can be parameterized as:

$$\mathbf{G}_1 = \begin{pmatrix} g_1 & g_2 & 0 \\ g_3 & g_4 & t_p \\ 2(g_1 - 1) & 2g_2 & 1 \end{pmatrix}; \quad g_1, g_2, g_3, g_4 \in \mathbb{F}_p \tag{5.28}$$

That the parameterization exists as above can be seen by looking at the hyperplane of intersection for the canonical translation, which according to equation (5.29) is now

$$\mathscr{H} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \tag{5.29}$$

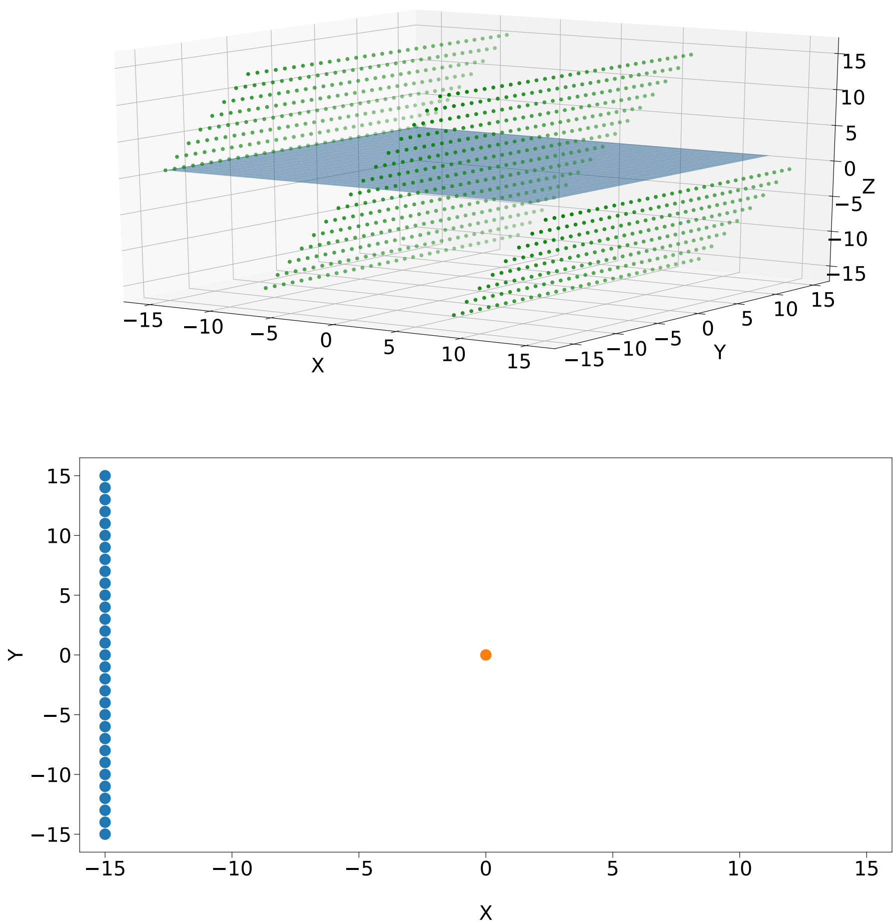


Figure 13: Hyperplane of intersection in the homogenous (top) and inhomogeneous (bottom) coordinates for prime p=31 for the reduced representation given by the canonical translation with $\mathbf{t} = \{1, 0\}^T$.

Then the matrix given by equation (5.28) solves equation (5.27).

This is a big group with $p^4(p-1)$ elements, where the factor p^4 which is subtracted represents singular matrices, which don't have inverses. This means that the group can be broken down into smaller subgroups since $|\mathbf{G}_1|$ has factors which can be broken down further, as explained by Lagrange's theorem for finite groups. These subgroups are expected to have aditional symmetries, so we define them in terms of the invariant structures. The first thing to notice is that the group can be broken down into transformations which keep the centre invariant and those which don't.

Definition 5.5. *Transformations keeping the centre invariant $\boldsymbol{G}_1(\boldsymbol{C})$*

The biggest non-trivial subgroup is the group of transformations which also keep the centre $\boldsymbol{C} = \{0,0,1\}^T$ invariant. We denote it as $\boldsymbol{G}_1(\boldsymbol{C})$, where $|\boldsymbol{G}_1(\boldsymbol{C})| = p^3(p-1)$, and the subgroup divides the bigger group into p co-sets. The action of this subgroup is especially clear by it's action on the line at infinity, since there is a tilt vector which allows for a shift of the line at infinity. The representation of the subgroup is given as:

$$\begin{pmatrix} G & \boldsymbol{0} \\ \boldsymbol{h}^T & 1 \end{pmatrix} \tag{5.30}$$

where we have used along with equation (5.27) *the condition*

$$\boldsymbol{G}_1(\boldsymbol{C})\boldsymbol{C} = \boldsymbol{C}$$

which implies $t_p = 0$ in equation (5.28).

A subgroup of $\mathbf{G}_1(\mathbf{C})$ is the group $\mathbf{G}_1(\mathbf{H}_c)$, which keeps the hyperplane of centres invariant. These are reflections around the space direction, where the mapping of points is dependent on the tilt of the hyperplane at infinity. Further subgroups of p and $p-1$ elements each exist, depending on whether the time coordinates are mapped to time coordinates or otherwise.

Definition 5.6. *Transformations keeping the direction of translation invariant*

These are transformations given by:

$$\boldsymbol{G}_1(\boldsymbol{H}_c) = \begin{pmatrix} g_1 & g_2 & 0 \\ 0 & 1 & 0 \\ 2(g_1-1) & 2g_2 & 1 \end{pmatrix} \tag{5.31}$$

Corollary 5.3. *The subgroup $\boldsymbol{SG}_1(\boldsymbol{H}_c)$ of matrices with determinant 1 are defined by the generator:*

$$\boldsymbol{SG}_1(\boldsymbol{H}_c) = \begin{pmatrix} 1 & g_2 & 0 \\ 0 & 1 & 0 \\ 0 & 2g_2 & 1 \end{pmatrix} \quad \rightarrow \quad \eth_1(\boldsymbol{H}_c) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

The group is then isomorphic to $\mathbb{Z} \pmod p$

A second subgroup of importance is the subgroup which keeps the hyperplane at infinity invariant. This is the subgroup $\mathbf{G}_1(\mathbf{H}^\infty)$ with $|\mathbf{G}_1(\mathbf{H}^\infty)| = p^2(p-1)$ elements. The action of this subgroup can be broken down into transformations keeping the centre invariant which belong to the group $AGL(n+1,p)$ and transformations which don't keep the centre invariant, which are translations along the hyperplane of intersection and are a subgroup of $\mathbf{T}(n+1,p)$.

Definition 5.7. *Transformations about the H^∞*

The group $\boldsymbol{G}_1(\boldsymbol{H}^\infty)$ has solutions for equation (5.27) *and for keeping the hyperplane at infinity invariant, thereby also solving*

$$\boldsymbol{G}_1(\boldsymbol{H}^\infty)^{-T}\boldsymbol{H}^\infty = \boldsymbol{H}^\infty$$

It has the subgroups $\boldsymbol{A} \in AGL(n+1,p)$ and $\boldsymbol{T}_p \in \boldsymbol{T}(n+1,p)$ of order $p(p-1)$ and p where we have $|\boldsymbol{G}_1(\boldsymbol{H}^\infty)| = |\boldsymbol{A}||\boldsymbol{T}_p|$. They divide the group action according to whether the centre is kept invariant or not. The representations are of the form:

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ g_1 & g_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \boldsymbol{T}_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t_p \\ 0 & 0 & 1 \end{pmatrix}, \quad g_1, g_2, t_p \in F_p \tag{5.32}$$

Corollary 5.4. *The group $\boldsymbol{T}_p$ of translations along $\mathscr{H}$ are generated by the element:*

$$\eth(\boldsymbol{T}_p) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.33}$$

Corollary 5.5. *The group $\boldsymbol{A}$ has two further subgroups $\boldsymbol{SA}$, $\boldsymbol{GA}$ which are distinguished by the action on the space coordinate in the affine plane. We have $\boldsymbol{SA}P(t,x,1) \longrightarrow P(t, gt+x, 1)$ and $\boldsymbol{GA}P(t,x,1) \longrightarrow P(t, gx, 1)$.*

Corollary 5.6. $|\boldsymbol{SA}| = p$ *and it is isomorphic to* $\mathbb{Z}/\ (\mathrm{mod}\ p)$. *It is the set of matrices with determinant 1. Therefore we have the generator:*

$$ð(SA) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Corollary 5.7. $|\boldsymbol{GA}| = p - 1$ *and it is isomorphic to* $\mathbb{Z}^*/\ (\mathrm{mod}\ p)$

So far we have looked at the two main structures which can be kept invariant. A common transformation might also exist however which keeps both the centre and the line at infiinity invariant. We find that there is 1 transformation $\in \mathrm{Aut}(Q, C) = \mathbf{L}$, which keeps (along with the hyperplane of intersection) the initial centre and the initial Minkowski quadric invariant.

Definition 5.8. *Quadric symmetry group*

The group $\mathfrak{G}_s$ *defined by the two elements* $\boldsymbol{I}$ *and the element given as:*

$$R_s = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.34}$$

is the group of transformations keeping the hyperplane of Intersection, the initial quadric, and the centre invariant.

Corollary 5.8. $\mathfrak{G}_s \subset \boldsymbol{L}$, *and in particular has the transformation which is a reflection along the space axis (with* $p - 1$ *scalings).*

A more general form of the above transformation is discussed in the section on general transformations. The important point here is that there exists another transformation apart from the identity which keeps the intial two objects invariant along with the hyperplane of intersection. Since the quadric and the centre also define the hyperplane at infinity, one finds that this transformation keeps the entire initial structure constant.

Our results on the general forms of type-1 transformations are presented below in the table 1.

Before we move on to other types of transformations, we look at some of the normal subgroups of the above groups.

Theorem 5.3. *Subgroup* $\boldsymbol{A}^+$

Group	Subgroup	Order	Form	Elements and Generators
$\mathbf{G}_1$	Itself	$p^4(p-1)$	$\begin{pmatrix} G & \mathbf{t}_p \\ \mathbf{h}^T & 1 \end{pmatrix}$	$G \in M\{2\times 2, F_p\}, \mathbf{h}^T = 2\{g_1 - 1, g_2\}$
	$\mathbf{G}_1(\mathbf{C})$	$p^3(p-1)$	$\begin{pmatrix} G & \mathbf{0} \\ \mathbf{h}^T & 1 \end{pmatrix}$	
	$\mathbf{G}_1(\mathbf{H}_c)$	$p(p-1)$	$\begin{pmatrix} G_t & \mathbf{0} \\ \mathbf{k}^T & 1 \end{pmatrix}$	$G_t = \begin{pmatrix} g_1 & g_2 \\ 0 & 1 \end{pmatrix}$
	$\mathbf{SG}_1(\mathbf{H}_c)$	p	$g_1 = 1$	$\eth^{H_C} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$
	$\mathbf{GG}_1(\mathbf{H}_c)$	$p-1$	$g_2 = 0$	
	$\mathbf{G}_1(\mathbf{H}^\infty)$	$p^2(p-1)$	$\begin{pmatrix} G_s & \mathbf{t}_p \\ \mathbf{0}^T & 1 \end{pmatrix}$	$G_s = \begin{pmatrix} 1 & 0 \\ g_1 & g_2 \end{pmatrix}$
	$\mathbf{T}_p$	p	$\begin{pmatrix} \mathbf{I} & \mathbf{t}_p \\ \mathbf{0}^T & 1 \end{pmatrix}$	$\mathbf{t}_p = \{0, t_p\}^T$, $\eth^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$
	$\mathbf{A}$	$p(p-1)$	$\begin{pmatrix} G_s & \mathbf{0} \\ \mathbf{0}^T & 1 \end{pmatrix}$	
	$\mathbf{SA}$	p	$g_2 = 1$	$\eth^A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
	$\mathbf{GA}$	$p-1$	$g_1 = 0$	
	$\mathfrak{G}_s$	2		$\mathbf{I}, R_s$

Table 1: Table of all important transformations of type-1 for $t_1 = 1, t_2 = 0$.

The subgroup $\boldsymbol{A}^+$ *generated by the group actions* $\boldsymbol{SA}$ *and* $\boldsymbol{GA}^+$*, where* $\boldsymbol{GA}^+$ *is the subgroup of* $(p-1)/2$ *elements of* $\boldsymbol{GA}$ *with a positive determinant is a normal subgroup of* $\boldsymbol{A}$.

Proof. Using Lagrange's Theorem we know that the order of a subgroup divides the order of it's group. Furthermore the subgroup partitions the group into co-sets. Now, for the case when $|G|/|H| = 2, \forall H \subset G$*, the group is divided into 2 co-sets one* *of which is the subgroup itself. This means that the right and left co-sets of the subgroup must be the same, and therefore the subgroup H is normal.*

□

The decomposition of any group into normal subgroups allows us to create what is known as a *subnormal series*. A series which is completely refined such that no other subgroups can be added to make a longer series is called a *composition series*. The composition series for a group G with maximal normal subgroup H and the identity element {e} is represented by: $\{e\} \lhd H \lhd G$. A fundamental result is that the factor groups in any two different composition series are isomorphic to each other. This is a consequence of the Jordan-Hölder-Schreier theorem. Further notes can be found in [Bau06].

Remark. *We have here the composition series:*

$$\{e\} \lhd \boldsymbol{GA}^{+} \lhd \boldsymbol{GA} \tag{5.35}$$

$$\{e\} \lhd \boldsymbol{A}^{+} \lhd \boldsymbol{A} \tag{5.36}$$

Similarly the group $\mathbf{G}_1^{+}(\mathbf{H}^{\infty})$ is the normal subgroup of $\mathbf{G}_1(\mathbf{H}^{\infty})$ and is defined by the group action of $\mathbf{A}^{+}$ with $\mathbf{T}_p$, such that we have the composition series:

$$\{e\} \lhd \mathbf{A}^{+} \lhd \mathbf{G}_1^{+}(\mathbf{H}^{\infty}) \lhd \mathbf{G}_1(\mathbf{H}^{\infty}) \tag{5.37}$$

Transformations of Type 2

These can be written similar to before as:

$$\mathbf{G}_2 = \begin{pmatrix} 1 & \alpha_1 & \alpha_2 \\ \beta_1 & \lambda_1 & \lambda_2 \\ 0 & 2\alpha_1 & 1+\alpha_2 \end{pmatrix} \tag{5.38}$$

The transformations allows us to find further symmetries. Due to the nature of the scaling however, we have the point at infinity which replaces the centre as a source of the lagest symmetries.

Definition 5.9. *Transformations about $\boldsymbol{P}^{\infty}$*

The group of transformations given by:

$$\boldsymbol{G}_2(\boldsymbol{P}^{\infty}) = \begin{pmatrix} 1 & \boldsymbol{k}' \\ \boldsymbol{0} & G \end{pmatrix} = \begin{pmatrix} 1 & g_3/2 & (g_4-1/)2 \\ 0 & g_1 & g_2 \\ 0 & g_3 & g_4 \end{pmatrix} \tag{5.39}$$

keeps the point at infinity invariant. In general the line at infinity is not kept invariant.

Corollary 5.9. $|\boldsymbol{G}_2(\boldsymbol{P}^\infty)| = p^3(p-1)$*, and the group is broken into p co-sets.*

Corollary 5.10. *The subgroup of $\boldsymbol{G}_2$ which don't keep the point at infinity invariant, are the same as the subgroup $\boldsymbol{A}$ given by equation* (5.32)*, and generated by group element.*

Therefore, the transformations are broken down into those which don't move the P^∞ and those that move it but only along L^∞. The latter is a mapping from the affine subspace $\mathfrak{U}^1$ to $\mathfrak{U}^0$.

The subgroup $\mathbf{G}_2(\mathbf{H}_c) \subset \mathbf{G}_2(\mathbf{P}^\infty)$ keeps the hyperplane of centres invariant.

Definition 5.10. *Transformations for the hyperplane of centres*

The subgroup $\boldsymbol{G}_2(\boldsymbol{H}_c)$ has $(p(p-1))$ elements and is generated by the group $\boldsymbol{SG}_2(\boldsymbol{H}_c)$ with determinant 1 and the generator:

$$\eth = \begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \tag{5.40}$$

where the inverse of 2 is defined on the field, and the group $\boldsymbol{GG}_2(\boldsymbol{H}_c)$ with (p-1) elements with the representation:

$$\boldsymbol{GG}_2(\boldsymbol{H}_c) = \begin{pmatrix} 1 & 0 & (g-1)/2 \\ 0 & 1 & 0 \\ 0 & 0 & g \end{pmatrix} \tag{5.41}$$

Remark. *The group $\boldsymbol{GG}_2(\boldsymbol{H}_c)$ doesn't keep the centre invariant and represents the translations along the hyperplane of intersection in the 3-dimensional vector space, such that for an affine point $\{a,b,1\}^T$, one has the transformed point $\{a+(g-1)/2, b, g\}$, such that the points on the hyperplane of intersection are mapped to the homogenous representation of another point on the hyperplane of intersection.*

Definition 5.11. *Transformations about H^∞*

Transformations keeping the hyperplane at infinity invariant are of the form:

$$\boldsymbol{G}_2(\boldsymbol{H}^\infty) = \begin{pmatrix} 1 & \boldsymbol{0} \\ \boldsymbol{t}_t & G_t \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ t & g_1 & g_2 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.42}$$

Corollary 5.11. $|\boldsymbol{G}_2(\boldsymbol{H}^\infty)| = p^2(p-1)$ *and is the same as the group $\boldsymbol{G}_1(\boldsymbol{H}^\infty)$, such that these are the only transformations we can have to keep the hyperplane at infinty invariant. This means $\boldsymbol{G}(\boldsymbol{H}^\infty) = \boldsymbol{G}_2(\boldsymbol{H}^\infty) \cup \boldsymbol{G}_1(\boldsymbol{H}^\infty) = \boldsymbol{G}_1(\boldsymbol{H}^\infty)$.*

Finally we look for another element which keeps also the quadric invariant. We find again another group of *order* 2 containing the identity and the Lorentz element which is a reflection along the x axis (time axis), given as:

$$R_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.43}$$

The two elements define again a group $\mathfrak{G}_t$, and one can combine this with $\mathfrak{G}_s$ given in equation (5.34) to get the maximal symmetry group.

Definition 5.12. *Maximal Symmetry group*

The group of order 3 containing the elements $\boldsymbol{I}, R_t, R_s$ is called the Maximal Symmetry group $\mathfrak{G}$, where $\mathfrak{G} = \mathfrak{G}_t \cup \mathfrak{G}_s$.

Corollary 5.12. *The group $\mathfrak{G} \subset \boldsymbol{L}$.*

Corollary 5.13. *The elements R_t and R_s generate the element $R_t R_s = R_s R_t$ which is a space-time reflection.*

We summarize the results for the generators of the groups here:

Definition 5.13. *Generators of the groups*

The group has generators for subgroups which are isomorphic to $\mathbb{Z} \pmod{p}$ of the form:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \tag{5.44}$$

5.2.2 Symmetries of the Union of Intersection planes

We have so far discussed the transformations which keep the $\mathscr{H}$ and $\mathbf{H}^\infty$ separately invariant and this is the group $\mathbf{G}(\mathbf{H}^\infty)$. There are two of the possible solutions that exist for keeping also the interunion invariant, and so we have $\mathbf{G}(\mathbf{H}^\infty) \subset \mathrm{Aut}(\mathscr{I}_U)$. However, these are not the only solutions. Another set of solutions inverts the structure, which is the mapping $\mathscr{H} \to \mathbf{H}^\infty$ and vice versa. Here we now talk about this inversion mapping, and the mathematical solution of this breakdown using the interunion representation itself is talked about later.

Group	Subgroup	Order	Form	Elements and Generators
$\mathbf{G}_2$	Itself	$p^4(p-1)$	$\begin{pmatrix} 1 & \mathbf{k}' \\ \mathbf{t}_t & G \end{pmatrix}$	$\mathbf{k}' = \frac{\{g_1, g_2-1\}}{2}, \mathbf{t}_t = \{t, 0\}^T$
	$\mathbf{G}_2(\mathbf{P}^\infty)$	$p^3(p-1)$	$\begin{pmatrix} 1 & \mathbf{k}' \\ \mathbf{0} & G \end{pmatrix}$	
	$\mathbf{G}_2(\mathbf{C})$	$p^2(p-1)$	$\begin{pmatrix} 1 & g_3/2 & 0 \\ t & g_1 & 0 \\ 0 & g_3 & 1 \end{pmatrix}$	
	$\mathbf{G}_2(\mathbf{H}_c)$	$p(p-1)$	$\begin{pmatrix} 1 & \mathbf{k}' \\ \mathbf{0} & G_s \end{pmatrix}$	
	$\mathbf{SG}_2(\mathbf{H}_c)$	p		$ð_2^{SG} = \begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$
	$\mathbf{GG}_2(\mathbf{H}_c)$	$p-1$	$\begin{pmatrix} 1 & 0 & (g-1)/2 \\ 0 & 1 & 0 \\ 0 & 0 & g \end{pmatrix}$	
	$\mathbf{G}_2(\mathbf{H}^\infty) = \mathbf{G}_1(\mathbf{H}^\infty)$	$p^2(p-1)$	$\begin{pmatrix} \mathbf{1} & \mathbf{0}^T \\ \mathbf{t}_t & G_t \end{pmatrix}$	
	$\mathbf{SA}$	p		$ð_2^{A} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
	$\mathbf{T}$	p		$ð_2^{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$
	$\mathbf{GA}$	$p-1$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & 1 \end{pmatrix}$	

Table 2: Table of all important transformations for type 2 for $t_1 = 1, t_2 = 0$.

Theorem 5.4. *Let $\mathbf{G}_D$ be the transformations keeping the interunion invariant. For the case of the interunion in 2-dimensions, the possible symmetries are given* *by:*

$$\boldsymbol{G}_D \mathscr{H} = \mathscr{H}, \boldsymbol{G}_D \boldsymbol{H}^\infty = \boldsymbol{H}^\infty \tag{5.45}$$

$$\boldsymbol{G}_D \mathscr{H} = \boldsymbol{H}^\infty, \boldsymbol{G}_D \boldsymbol{H}^\infty = \mathscr{H} \tag{5.46}$$

Proof. The general form of invariance requires the interunion to be kept invariant. This is the equation for the representation of the interunion and is given as:

$$\boldsymbol{G}_D^{-T} \mathscr{I}_U \boldsymbol{G}^{-1} = \mathscr{I}_U \tag{5.47}$$

However the above can only have two solutions. Since the interunion is the disjoint union of the hyperplane of intersection and the line at infinity, one has either the mappings of the lines to themselves, or the mappings where the lines are exchanged.

□

The fact that the solution dissolves into these two transformations is revisited later for general transformations. We arrive now at a fundamental property of these transformations.

Theorem 5.5. *The group $\boldsymbol{G}(\boldsymbol{H}^\infty)$ is a normal subgroup of $\boldsymbol{G}_D$.*

Proof. Let the transformations which exchange the point set of the hyperplanes be given as $\boldsymbol{G}_e$ such that $\boldsymbol{G}_D = \boldsymbol{G}(\boldsymbol{H}^\infty) \cup \boldsymbol{G}_e$. The transformations $\boldsymbol{G}_e$ however do not form a group, but instead belong to the Co-sets, such that $|\boldsymbol{G}_D| = |Co - sets||\boldsymbol{G}(\boldsymbol{H}^\infty)|$, where $|Co - sets| = 2$, such that we have $|\boldsymbol{G}_D| = 2|\boldsymbol{G}(\boldsymbol{H}^\infty)|$. Therefore since we have now two co-sets and one of them is $\boldsymbol{G}(\boldsymbol{H}^\infty)$ we have again that $\boldsymbol{G}(\boldsymbol{H}^\infty)$ is a normal subgroup.

□

Remark. *We have the quotient group $F = \boldsymbol{G}_D / \boldsymbol{G}(\boldsymbol{H}^\infty)$, and the composition series:*

$$\{e\} \lhd \boldsymbol{G}(\boldsymbol{H}^\infty) \lhd \boldsymbol{G}_D \tag{5.48}$$

We now look at the representation of the co-set.

Definition 5.14. *Exchange Transformation for hyperplanes*

The set of exhange transformations of type 1 are represented by:

$$\boldsymbol{G}_{e_1} = \begin{pmatrix} -1 & 0 & 0 \\ g_1 & g_2 & g_3 \\ -2 & 0 & 1 \end{pmatrix} \tag{5.49}$$

For the case of reduced transformations transformations of type 2 are just the scalings of type 1. So we have:

$$\boldsymbol{G}_{e_2} = -1\boldsymbol{G}_{e_1} \tag{5.50}$$

We therefore drop the numbering for this case.

It is possible to have transformations which are not scaled, and these are discussed later.

As we can see the representation has a tilt vector, which is now required due to the condition that we change the line at infinity. This is trivial for our case, since the last row is the representation of the hyperplane of intersection (transposed) while the first row is the line perpendicular to the movement of translation, which in this case is simply the space (y) axis. This allows us to write the transformations as:

$$\mathbf{G}_e = \begin{pmatrix} \ldots (H_t)^T \ldots \\ \ldots L^T \ldots \\ \ldots \mathscr{H}^T \ldots \end{pmatrix} \tag{5.51}$$

The above parametrisation then allows us to define the action in terms of incidence relations. We can parametrize subsets according to the line L^T. For instance a representation is given by the line through the centre such that $L = \{g_1, g_2, 0\}$. This is the subset without any in plane translations with $p(p-1)$ elements.

The table 3 has the representations for the transformations which invert the line at infinity and the hyperplane of intersection.

Group	Subgroup	Number of elements	Form	Notes
$\mathbf{G}_e$	Itself	$p^2(p-1)$	$\begin{pmatrix} -1 & 0 & 0 \\ g_1 & g_2 & g_3 \\ -2 & 0 & 1 \end{pmatrix}$	
	$\mathbf{G}_e^T$	p	$\begin{pmatrix} \mathbf{I} & t_s \\ C & -1 \end{pmatrix}$	C={2,0}
	$\mathbf{G}_e^L$	$p(p-1)$	$\begin{pmatrix} G_s & \mathbf{0}^T \\ C & -1 \end{pmatrix}$	

Table 3: Table of all important transformations for inversion of hyperplanes $(t_1 = 1, t_2 = 0)$

5.3 Successive Translations

So far we have defined the hyperplanes and interunion in terms of the initial centre and a translated one. These were defined in (5.23). However, a more generalized case also exists where in one can have hyperplanes between any two arbitrary centres. In particular, as a case of importance is the case when the two points are successively translated. Physically this corresponds to the case when the acceleration is 0 and the momentum generates the translations such that the set of centres lies on the lines generated by this momentum.

We now therefore shortly introduce the case of successive translations.

Definition 5.15. *Successive Translation*

The case of similar successive translations allows for a generation of translations from the initial choice of the translation vector. For a vector $\boldsymbol{t}$*, multiple actions of the vector will generate successive translations in the same direction. The translation matrix for an arbitrary translation generated by such a vector is then represented as:*

$$\boldsymbol{T}_\gamma = \begin{pmatrix} \boldsymbol{I} & \gamma \boldsymbol{t} \\ 0 & 1 \end{pmatrix}, \textit{where } \gamma = 0, 1.... \tag{5.52}$$

Owing to these one has then the corresponding interunion (due to $(\gamma+1)^{th}$ *translation and* $(\gamma)^{th}$ *translation):*

$$\mathscr{I}_U^\gamma = \begin{pmatrix} \boldsymbol{0}_{d\times d} & -G_{min}\boldsymbol{t} \\ -(G_{min}\boldsymbol{t})^T & (2\gamma+1)t_m \end{pmatrix} \tag{5.53}$$

For points in the affine-plane, one has the hyperplane of intersection, and for the successive translation cases, one finds a family of such hyperplanes, given by:

$$\mathscr{H}^\gamma = \begin{pmatrix} -2G_{min}\boldsymbol{t} \\ (2\gamma+1)t_m \end{pmatrix} \tag{5.54}$$

For $\gamma = 0$ we get back the results which we have discussed.

Theorem 5.6. *For any arbitrary* γ*, the transformations* $\boldsymbol{G}_\gamma$ *keeping the hyperplane of intersection or the interunion invariant are given by,*

$$\boldsymbol{G}_\gamma = \boldsymbol{T}_\gamma^{-T}\boldsymbol{G}\boldsymbol{T}_\gamma^{-1} \tag{5.55}$$

where $\boldsymbol{G}$ *are the initial transformations which keep* $\mathscr{H}^0$ *invariant, as we have discussed before in section 5.2.*

Affine points of intersection of quadrics

Instead of the family of hyperplanes of intersection given by the points of intersection, one can talk also about two more hyperplanes which exist for successive translations. These are the hyperplanes on which two successive intersection points shall lie. In other words, this is the trajectory of the actual affine intersection points given by equation (5.13). Let us now see what this hyper-plane looks like.

First, we find the affine points of intersection given by (5.54) for $i = 0$. These are two points, written as: $\mathbf{p}^j = \{p_1^j, p_2^j, 1\}^T, j \in \{1,2\}$, where the two p_1, p_2 are given by:

$$p_1 = \frac{2t_2 p_2 - t_m}{2t_1}, \quad (p_2)^2 = (p_1)^2 - 1 \tag{5.56}$$

These affine points can be generalized for any successive intersections by:

$$p_1 = \frac{2t_2 p_2 - (2i+1)t_m}{2t_1}, \quad (p_2)^2 = (p_1)^2 - 1 + i(i+1)t_m \tag{5.57}$$

It is trivial then that the successive points of intersection are actually just the translation of initial points of intersection. This means that one can write these points as:

$$\mathbf{p}^j(i) = \mathbf{p}^j(0) + i\mathbf{t} \tag{5.58}$$

Therefore one now has a trajectory of these neighbourhood points of the centre, at least in the affine plane (a more general discussion willl follow). One can find this trajectory by the *cross-product* of any two points of intersection. This is similar to finding two hyperplanes dual to the initial points of intersection, where these hyperplanes describe the trajectory of these points.

The hyperplanes are then given as $\mathscr{T}^j$ representing the trajectory of intersection points. We use equation (2.13) for the cross-product and equation (5.58) with $i = 1$ to get

$$\mathscr{T}^j = \det \begin{pmatrix} p^j(0) \\ p^j(0) + \mathbf{t} \\ \hat{e} \end{pmatrix}$$

such that we have

$$\mathscr{T}^j = \begin{pmatrix} -t_2 \\ t_1 \\ (2p_2^j - t_2)\frac{t_m}{2t_1} \end{pmatrix}, \quad j = \{1,2\} \tag{5.59}$$

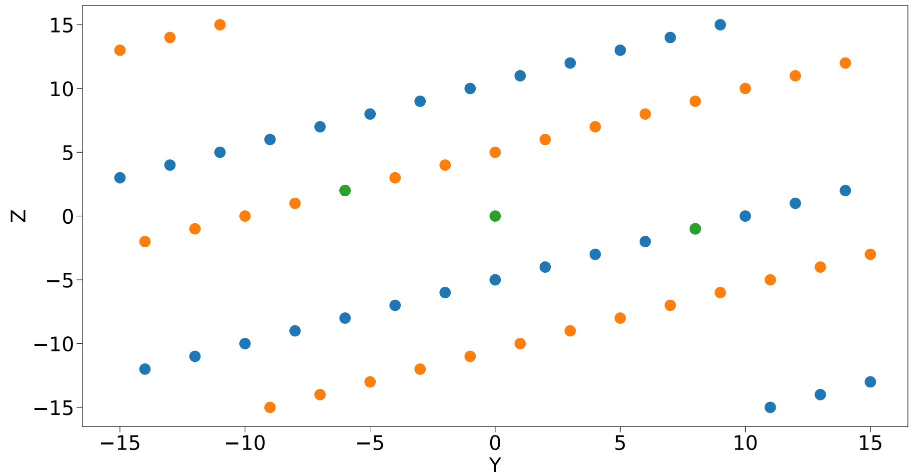


Figure 14: The two trajectories for the two affine points of intersection given by equation (5.59). Plotted here are the two points and the centre in green, and the two trajectories on which the points lie seperately. The affine parallel nature is clear from the graph.

Figure 14 shows the two trajectories in the affine plane.

The transformations $G(\mathscr{T})$ which keep both of the trajectories simultaneously invariant are then given by a subset of the affine transformations such that we have

$$G(\mathscr{T})^{-T}\mathscr{T}^j = \mathscr{T}^j$$

and we can parametrize it as

$$G(\mathscr{T}) = \begin{pmatrix} g_1 & \frac{t_1(g_2-1)}{t_2} & 0 \\ \frac{t_2(g_1-1)}{t_1} & g_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_1, g_2 \in \mathbb{F}_p \tag{5.60}$$

5.4 Arbitrary Dimensions

The above 2-dimensional calculation can be extended to arbitrary dimensions. The translated quadric is given by (5.1). For points $P = \{\mathbf{p}_{\mathrm{d}\times 1}, p_{d+1}\}^T$ in a d-dimensional projective space, one can solve for both the translated and the original quadric to get the following equation:

$$-2\mathbf{p}^T(G_{\min}t)p_{d+1} + t_m p_{d+1}^2 = 0 \tag{5.61}$$

This equation is the generalisation of the interunion, given by:

$$\begin{pmatrix} \mathbf{0}_{d\times d} & -G_{\min}\mathbf{t} \\ -(G_{\min}\mathbf{t})^T & t_m \end{pmatrix} \tag{5.62}$$

where both the $G_{\min}$ and $\mathbf{t}$ are the d-dimensional generalisations, as given in the introduction. The group of transformations keeping the interunion invariant (excluding the co-set of exchange transformations) are then a d^2-1 parameter group given by the representation:

$$G = \begin{pmatrix} G_1 & G_2 \\ 0 & 1 \end{pmatrix} \tag{5.63}$$

and G_1 and G_2 are given by:

$$G_1^T(-G_{\min}t) = (-G_{\min}t) \tag{5.64}$$

$$-G_2^T(G_{\min}t) = (G_{\min}t)^T G_2 \tag{5.65}$$

The minus sign in (5.64) represents that G_1 keeps the hyper-plane in the affine plane invariant.

As an example, for the 3-dimensional group one has:

$$G_1 = \begin{pmatrix} g_1 & g_2 & \frac{t_1(g_1-1)-t_2g_2}{t_3} \\ \frac{t_2(g_3-1)+t_3g_4}{t_1} & g_3 & g_4 \\ g_5 & \frac{t_3(1-g_6)+t_1g_5}{t_2} & g_6 \end{pmatrix}, \quad G_2 = \begin{pmatrix} \frac{g_7t_2+g_8t_3}{t_1} \\ g_7 \\ g_8 \end{pmatrix} \tag{5.66}$$

The hyperplane of intersection in 3-dimensions can be written as:

$$\mathscr{H} = \begin{pmatrix} 2t_1 \\ -2t_2 \\ -2t_3 \\ t_m \end{pmatrix}, \quad t_m = t_3^2 + t_2^2 - t_1^2 \tag{5.67}$$

with the reduced representation as:

$$\begin{pmatrix} \mathscr{H}_a \\ 1 \end{pmatrix} \tag{5.68}$$

For type-1 transformations[29] of the form,

$$\mathbf{G}_1 = \begin{pmatrix} \Lambda & \alpha \\ \beta & 1 \end{pmatrix} \tag{5.69}$$

we have the parameterisation with the equations

$$\Lambda^T \mathscr{H}_a + \beta^T = \mathscr{H}_a \tag{5.70}$$

$$\alpha^T \mathscr{H}_a = 0 \tag{5.71}$$

The two main subgroups as given by the action on the centre and the plane at infinity, such that:

$$\mathbf{G}_1(\mathbf{C}) = \begin{pmatrix} \Lambda & 0 \\ (\Lambda - \mathbf{I})\mathscr{H}_a^T & 1 \end{pmatrix} \tag{5.72}$$

$$\mathbf{G}_1(\mathbf{H}^\infty) = \begin{pmatrix} \Lambda & \alpha \\ \mathbf{0} & 1 \end{pmatrix} \tag{5.73}$$

with α solving equation (5.71), the set of points which lie on the hyperplane, and Λ keeping the reduced $\mathscr{H}_a$ invariant.

[29]Type-2 transformations follow a similar parameterization and are therefore omitted here.

6 Symmetries for Generalized Translations

Time is an illusion. Lunchtime doubly so.

Douglas Adams, The Hitchhiker's guide to the galaxy

Now that we have some results with respect to both Lorentz and gauge transformations, we can analyse how the gauge group acts on the geometrical objects we are concerned with, in the generalized translations. To this end, first let us re-count all the main objects of study we have so far.
The main objects are the initial Minkowski quadric, the transformed quadric, the interunion, the intersection points and their respective hyperplanes, and the translation of the intersection points (which is parallel to the translation vector). We are then concerned with the subgroups of the larger groups we found before, namely the group which keeps the interunion invariant, and the group which keeps the hyperplane of intersection invariant. In the end we will also look at general projectivities and how they act on multiple objects.

The quadrics are given by:

$$\mathbf{Q}_{\min} = \begin{pmatrix} -1 & \mathbf{0} \\ \mathbf{0}^T & \mathbf{I}_{2\times 2} \end{pmatrix}, \mathbf{Q}_{\text{trans}} = \begin{pmatrix} G_{\min} & -G_{\min}\mathbf{t} \\ -(G_{\min}\mathbf{t})^T & 1 + t_m \end{pmatrix} \tag{6.1}$$

The interunion, the hyperplane of affine points of intersection, and the two quadric points at infinity are given by :

$$\mathscr{I}_{\mathrm{U}} = \begin{pmatrix} \mathbf{0}_{2\times 2} & -G_{\min}\mathbf{t} \\ -(G_{\min}\mathbf{t})^T & t_m \end{pmatrix}, \quad \mathscr{H} = \begin{pmatrix} 2t_1 \\ -2t_2 \\ t_m \end{pmatrix}, \quad \mathbf{P}_Q^{\text{inf}} = \{1, \pm 1, 0\}^T \tag{6.2}$$

The direction of movement is defined by and the hyperplanes of the translation are given by:

$$\mathbf{T} = \begin{pmatrix} \mathbf{I}_{2\times 2} & \mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix}, \quad \mathbf{t} = \{t_1, t_2\}^T, \quad \mathscr{T}^j = \begin{pmatrix} -t_2 \\ t_1 \\ (2p_2^j - t_2)\frac{t_m}{2t_1} \end{pmatrix} \tag{6.3}$$

One can then use the two centres to get the hyperplane on which these and the successively translated centres shall lie. This hyperplane and the first two centers are given as:

$$\mathbf{C}_{\min} = \{0,0,1\}^T, \quad \mathbf{C}_{\text{trans}} = \{t_1, t_2, 1\}^T, \quad \mathbf{H}_C = \{-t_2, t_1, 0\} \tag{4.4}$$

As can be seen $\mathbf{H}_C$ and $\mathscr{T}$ are parallel to each other in the affine plane. They have a point of intersection only at infinity whch is given by the cross-product $\mathbf{H}_C \times \mathscr{T}$.

6.1 Parametrization of the 2-dimensional generalized translations

These representations follow the same naming as the case of reduced transformations, and therefore we shall now use the naming convention from before to refer to the bigger groups. The representations below include $\mathbf{t}$ as a parameter, such that the group of transformations is much bigger than the groups we have talked about so far. Therefore, the groups we have now represent transformations for all possible hyperplanes between the entire affine plane as a choice for the second centre.

6.1.1 Hyperplane of intersection

Type-1 representations

The first transformations are those we call type 1 transformations which involve the last element of the matrix to be set to 1. We represent these with a '1' in the subscript. We have,

$$\mathbf{G}_1 = \begin{pmatrix} G & \mathbf{t}_p \\ (\mathscr{H}_a)^T(\mathbf{I} - G) & 1 \end{pmatrix}; \quad \mathbf{I}, G \in M(2 \times 2, p) \tag{6.4}$$

Here we have redefined $\mathscr{H}$ by a reduced representation $\mathscr{H} \to \{\mathscr{H}_a, 1\}^T$, such that,

$$\begin{pmatrix} \mathscr{H}_a \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{-G_{\min}\mathbf{t}}{t_m} \\ 1 \end{pmatrix} \tag{6.5}$$

The further parameterization allows us to break down this group into a subgroup, which we call $\mathbf{T}_p$, given as:

$$\mathbf{T}_p = \begin{pmatrix} 1 & 0 & t_2 g_5 \\ 0 & 1 & t_1 g_5 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore, we have $\mathbf{T}_p \in \mathbf{T}$. These are precisely the translations along the hyperplane of intersection in the affine plane, thereby keeping it invariant. One can see in the vector-space, that the dot product of the translation vector with the hyperplane is 0. These are $|\mathbf{T}_p| = p$ translations (for particular t_1, t_2), with additional $(p^2 - 1) = (p-1)(p+1)$ degrees of freedom coming from the translations. These translations are generated by the element $g_5 = 1$, such that we have now the generator:

$$(\mathbf{T}_p) = \begin{pmatrix} 1 & 0 & t_2 \\ 0 & 1 & t_1 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.6}$$

Also using Lagrange's theoreom for finite groups, these translations then parition the general group into co-sets.

A bigger, special subgroup of $\mathbf{G}_1$ is the group of in-plane projectivities, where the *tilt* is $\mathbf{0}$ which is defined by the action of the representative matrix G by,

$$G^T \mathscr{H}_a = \mathscr{H}_a \tag{6.7}$$

such that we have:

$$\mathbf{G}_1(\mathbf{H}^\infty) = \begin{pmatrix} 1 + g_1 t_2 & t_2 g_2 & t_2 g_5 \\ t_1 g_1 & 1 + t_1 g_2 & t_1 g_5 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.8}$$

such that $\mathbf{T}_p \in \mathbf{G}_1(\mathbf{H}^\infty)$, and the subgroup $\mathbf{A} \in \mathbf{G}_1(\mathbf{H}^\infty)$, where

$$\mathbf{A} = \begin{pmatrix} 1 + g_1 t_2 & t_2 g_2 & 0 \\ t_1 g_1 & 1 + t_1 g_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.9}$$

The division into $\mathbf{T}_p$ and $\mathbf{A}$, can be shown to make the entire group, such that:

$$\forall T \in \mathbf{T}_p, \forall A \in \mathbf{A}; \quad G_1, G_2 \in G_1(\mathbf{H}^\infty)$$

$$\implies \quad T * A = G_1, \tag{6.10}$$

$$A * T = G_2 \tag{6.11}$$

In general $G_1 \neq G_2$. Furthermore, since both $\mathbf{A}$ and $\mathbf{T}_p$ are sub-groups, their orders must divide the order of the bigger group. In this case, we have that $|\mathbf{G}_1(\mathbf{H}^\infty)| = p^2(p-1)$ elements, where p of these elements are in $\mathbf{T}_p$ and $p(p-1)$ in $\mathbf{A}$.

Therefore, we have the property that the subgroups $\mathbf{A}$ and $\mathbf{T}_p$ are the corresponding co-sets of each other, with:

$$|\mathbf{G}_1(\mathbf{H}^\infty)| = |\mathbf{T}_p||\mathbf{A}| \tag{6.12}$$

Like before in table 1, $\mathbf{A}$ is divided further into the subgroup $\mathbf{GA}$ and the set of transformations $\mathbf{SA}$ such that for the representation one has $\det(\mathbf{SA}) = 1$ with:

$$\mathbf{GA} = \begin{pmatrix} \lambda & \frac{t_2(\lambda-1)}{t_1} & 0 \\ \frac{t_1(\lambda-1)}{t_2} & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{SA} = \begin{pmatrix} \lambda & -\frac{t_2(\lambda+1)}{t_1} & 0 \\ \frac{t_1(\lambda-1)}{t_2} & -\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.13}$$

and the order,

$$|\mathbf{A}| = |\mathbf{GA}||\mathbf{SA}| \tag{6.14}$$

$$|\mathbf{GA}| = p-1, \quad |\mathbf{SA}| = p \tag{6.15}$$

Remark. *Even though* $\boldsymbol{SA}$ *has p elements, the mapping* $\phi : \boldsymbol{SA} \to \mathbb{Z} \pmod p$ *is not a group homomorphism since* ker_ϕ *doesn't contain the identity matrix* $\boldsymbol{I}$*. Only in the case of either of* t_1, t_2 *being 0 is the mapping a group homomorphism. However we have ignored that case here due to the difference in matrix representation.*

The sub-groups becomes clearer when one considers the action on the centre. In that case there are no translations allowed, and the group shrinks down to the 4 parameter group $\mathbf{G}_1(\mathbf{C})$, such that we now have the representation G, and its action on the reduced hyperplane $\mathscr{H}_a$.

$\mathbf{G}_1(\mathbf{C})$ has a special sub-group as well, which keeps the direction of translation invariant. This is the two-parameter group, and we parametrize is by first writing the hyperplane as:

$$H_c = \begin{pmatrix} H \\ 0 \end{pmatrix} \tag{6.16}$$

Therefore we have for $\mathbf{G}_1(\mathbf{H}_C)$ the representative equation:

$$G^T H = H \tag{6.17}$$

and we can parametrize it as

$$\mathbf{G}_1(\mathbf{H}_C) = \begin{pmatrix} 1+t_1g_1 & t_1g_2 & 0 \\ t_2g_1 & 1+t_2g_2 & 0 \\ 2g_1 & 2g_2 & 1 \end{pmatrix} \tag{6.18}$$

Remark. *$|\boldsymbol{G}_1(\boldsymbol{H}_C)| = p(p-1) = |\boldsymbol{A}|$ and therefore there exists an isomorphism $\Phi : \boldsymbol{G}_1(\boldsymbol{H}_C) \to \boldsymbol{A}$.*

The isomorphism allows the division again into special and general parts with det = 1 or otherwise.

Remark. *The group $\boldsymbol{G}_1(\boldsymbol{H}_C)$ may also be understood by writing the two subgroups with $p-1$ elements each, which are defined by their action on arbitrary points, such that we have for the two columns χ_t and χ_s:*

$$P(p_1, p_2, p_3) \to P + \chi_t p_1, \tag{6.19}$$

$$P(p_1, p_2, p_3) \to P + \chi_s p_2 \tag{6.20}$$

where χ_t and χ_s are the columns for $g_1 = 0$ and $g_2 = 0$ respectively.

Finally, we have also the quadric symmetry elements, $\mathfrak{G}_1 \subset \mathbf{G}_1(\mathbf{H}^\infty)$ with the two elements, the identity and the Lorentz element:

$$\mathfrak{L} = \begin{pmatrix} -\frac{t_e}{t_m} & \frac{2t_1t_2}{t_m} & 0 \\ -\frac{2t_1t_2}{t_m} & \frac{t_e}{t_m} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad t_e = t_1^2 + t_2^2 \tag{6.21}$$

Remark. *The transformation $\mathfrak{L}$ can be explicitly shown to be a Lorentz transformation in two dimensions by writing $\frac{t_e}{t_m}$ as $\frac{\lambda+1/\lambda}{2}$, giving $\lambda = \frac{t_2+t_1}{t_2-t_1}, \frac{t_2-t_1}{t_2+t_1}$.*

Type-2 representations

These are transformations of the form $\mathbf{G}_2$ with the first matrix element as 1, so $\mathbf{G}_2 = \begin{pmatrix} 1 & \alpha \\ \beta^T & \lambda \end{pmatrix}$.

The first important subgroups are the 4 and the 3 parameter groups keeping the point at infinity ($\mathbf{G}_2(P^\infty)$) and the affine center invariant.

Two other important subgroup of these transformations are the transformations which keep either the hyperplane of centres or the line at infinity invariant, of the form:

$$\mathbf{G}_2(\mathbf{H}_C) = \begin{pmatrix} 1 & \lambda_1 t_1 & \lambda_2 t_1 \\ 0 & 1+t_2\lambda_1 & \lambda_2 t_2 \\ 0 & 2\lambda_1 & 1+2\lambda_2 \end{pmatrix}, \mathbf{G}_2(\mathbf{H}^\infty) = \begin{pmatrix} 1 & \lambda_1 t_2 & \lambda_2 t_2 \\ 0 & 1+\lambda_1 t_1 & \lambda_2 t_1 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.22}$$

We see now that $\mathbf{G}_2(\mathbf{H}^\infty) \in \mathbf{G}_1(\mathbf{H}^\infty)$, and the subgroup can be seen as those transformations which aditionally keep the point $P^\infty = \{1,0,0\}^T$ invariant. The symmetry between this point and the center with respect to the action of the two types of groups can be seen here, such that now the point at infinity defines a bigger subgroup, and is necessarily kept invariant along with the line at infinity. On the other hand $\mathbf{G}_2(\mathbf{H}_C)$ is not a subgroup of $\mathbf{G}_2(\mathbf{C})$ such that one can define the action of this subgroup in terms of the new centre by $\mathbf{G}_2(\mathbf{H}_C)\mathbf{C} \to \mathbf{C}_{\text{new}}$, where:

$$\mathbf{C}_{\text{new}} = \begin{pmatrix} t_1 \\ t_2 \\ \frac{1+2\lambda_2}{\lambda_2} \end{pmatrix} = \begin{pmatrix} \epsilon t_1 \\ \epsilon t_2 \\ 1 \end{pmatrix}; \quad \epsilon = \frac{1+2\lambda_2}{\lambda_2} \in F_p \tag{6.23}$$

Another important subgroup is the group with the first column and first row set as (1,0,0). These transformations keep invariant the line {1,0,0} which has points in the affine plane of the form $\{0,a,1\}^T$ for some $a \in F_p$. This is also clearly the case when $t_1 = 0$, and we move in the space direction.

Two special case of these transformations are then the one-parameter group and the co-set, given as:

$$\mathbf{G}^{\text{sym}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & \frac{t_m(\lambda-1)}{2t_2} \\ 0 & \frac{2t_2(\lambda-1)}{t_m} & \lambda \end{pmatrix}, \quad \mathbf{G}^{\text{Imp}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & -\frac{t_m(\lambda+1)}{2t_2} \\ 0 & \frac{2t_2(\lambda-1)}{t_m} & -\lambda \end{pmatrix} \tag{6.24}$$

These transformations have $p-1$ and p elements, with the latter having $\det \mathbf{G}^{\text{Imp}} = -1$ (Imp is improper here, notation taken from [Pes19]).

Moving on a light cone

A special case which has so far been omitted is the case of the light cone. This is represented by the translation vector where $|t_1| = |t_2| = t$. There are no points of intersection in the affine case, even for really large values of translation. The translated quadric "intersects" the initial quadric only at one point which is at infinity. This point is either of two points at infinity defined before, depending on in which direction one (since $t_2 = \pm t_1$) moves.

6.1.2 Union of Intersection Planes

First we now show the solutions which can exist for the invariance of the interunion.

Theorem 6.1. *The invariance of the interunion has two solutions, the case where the hyperplanes are kept simultaneously invariant, and the case when we exchange them.*

Proof. *We show here for type 1 transformations.*

$$\boldsymbol{G}^T \mathscr{I}_U \boldsymbol{G} = \mathscr{I}_U$$

We write $\boldsymbol{G}$ *and* $\mathscr{I}_U$ *as:*

$$\boldsymbol{G} = \begin{pmatrix} \lambda_{2\times 2} & \alpha_{2\times 1} \\ \beta_{1\times 2} & 1 \end{pmatrix}, \quad \mathscr{I}_U = \begin{pmatrix} 0 & -G_{min}\boldsymbol{t} \\ -(G_{min}\boldsymbol{t})^T & t_m \end{pmatrix}$$

This gives us:

$$-\lambda^T G_{min}\boldsymbol{t}\beta - \beta^T (G_{min}\boldsymbol{t})^T \lambda + t_m \beta^T \beta = 0 \tag{6.25}$$

$$-\lambda^T G_{min}\boldsymbol{t} - \beta^T (G_{min}\boldsymbol{t})^T \alpha + t_m \beta^T = -G_{min}\boldsymbol{t} \tag{6.26}$$

and further,

$$\alpha G_{min}\boldsymbol{t} = -(G_{min}\boldsymbol{t})^T \alpha \tag{6.27}$$

The solution for α *comes straight from* (6.27)*. We use that in* (6.26) *to further get:*

$$\beta = (G_{min}\boldsymbol{t})^T (\lambda - \boldsymbol{I})/t_m \tag{6.28}$$

Using this in (6.25)*, we get:*

$$\lambda^T (G_{min}\boldsymbol{t})(G_{min}\boldsymbol{t})^T \lambda = (G_{min}\boldsymbol{t})(G_{min}\boldsymbol{t})^T \tag{6.29}$$

The solution for the final equation has two possible parameterizations, and these correspond to the two sets of solutions (simultaneous invariance and inversion) that we have talked about above.

□

The first transformations are those which keep both the hyperplane and the line at infinity invariant simultaneously. These are the transformations $\mathbf{G}_1(\mathbf{H}^\infty)$, given by equation (6.8) with the special subgroup $\mathbf{G}_2(\mathbf{H}^\infty)$ which also keeps the P^∞ invariant.

The second transformation can be found by exchanging the affine points with the points at infinity. These belong to a general class of transformations exchanging two lines.

Definition 6.1. *Exchange Transformations*

For any general line l and the line at infinity l^{∞}, and a general projectivity P, we have:

$$P^{-T}l = l^{\infty}, P^{-T}l^{\infty} = l \tag{6.30}$$

For a non-singular transformation this implies that $P^T P^T l^{\infty} = l^{\infty}$. This means that exhanging the lines twices gives us back the same lines.

Now for a moment, let's forget about the general line, and just focus on the transformations which give us the line at infinity back on simultaneous application. This representation can be parameterised as (see Appendix):

$$\boldsymbol{P}_{exch} = \begin{pmatrix} g_1 & -\frac{(g_2+1)g_5}{g_4} & g_3 \\ -\frac{g_4(g_1+1)}{g_5} & g_2 & -\frac{g_3 g_4}{g_5} \\ g_4 & g_5 & 1 \end{pmatrix}; g_i \in F_p \tag{6.31}$$

In particular any $\boldsymbol{G}_e < P_{exch}$. Then $\boldsymbol{G}_e$ can be found by using the condition: $P^T l^{\infty} = l$, and for $l = \{l_1, l_2, 1\}^T$, one has:

$$g_4 = l_1, g_5 = l_2$$

Then $\mathbf{G}_{e_1}$ belongs to those transformations which exchange the elements of two lines. such that we have,

$$\mathbf{G}_{e_1} = \begin{pmatrix} G & \mathbf{t}_p \\ \mathscr{H}_a^T & 1 \end{pmatrix} \tag{6.32}$$

where G transforms the subspace:

$$G^T \mathscr{H}_a = -\mathscr{H}_a \tag{6.33}$$

There are again transformations of the form with a trivial solution for equation (6.33) such that we have the subgroup

$$\begin{pmatrix} \mathbf{I} & \mathbf{t}_p \\ \mathscr{H}_a^T & 1 \end{pmatrix}$$

The second subgroup is the non-trivial solution but which keeps the centre invariant such that we have the form

$$\begin{pmatrix} G & \mathbf{0} \\ \mathscr{H}_a^T & 1 \end{pmatrix}$$

The final transformations are then type-2 transformations with the first element as 1, which exchanges again the affine and infinity points, and are given as:

$$\mathbf{G}_{e_2} = \begin{pmatrix} 1 & g_1 t_2 & g_2 t_2 \\ \frac{t_1(t_m+1)}{t_2} & g_1 t_1 - t_m & g_2 t_1 + \frac{t_m^2-1}{2t_2} \\ 2t_1 & -2t_2 & t_m \end{pmatrix} \tag{6.34}$$

The effect of using a general element of $\mathbf{G}_{e_1}$ is shown in figure 15. The point at $z = 0$ are mapped back to their counterparts on the line at infinity (since we are using homogeneous coordinates), and therefore the line at infinity must not be confused for a plane.

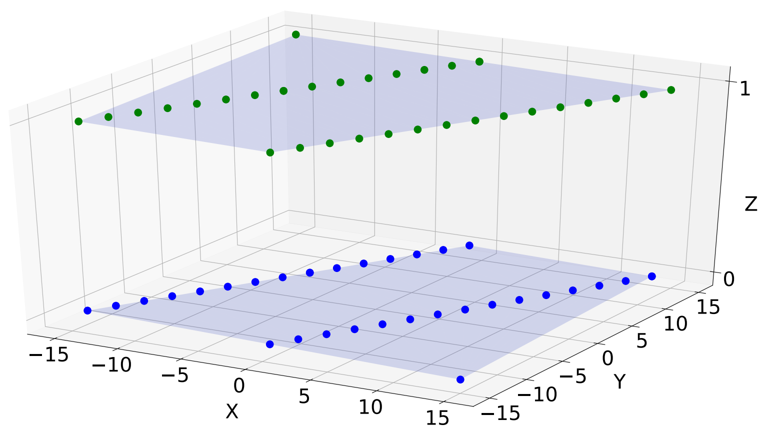


Figure 15: The exchange transformation for an element defined by equation (6.32). Points in green represent the original affine points in the hyperplane of intersection. Blue points are the points after the action of an element of exchange transformation. Shown here are results for p=31, $\mathbf{t} = \{2, 1\}^T$.

This concludes our work on all symmetries of the hyperplane of intersection, and we present our final results in table 4.

6.2 Enumeration of symmetry groups of projective spaces

One of the most useful aspects of working in finite spaces is that everything can be enumerated.[30] This allows us to define isomorphisms and mappings

[30]This holds true in principle. However computation problems might arise for higher order calculations.

Group/Set	Subgroup	Order	Invariant structures	Notes
$\mathbf{G}_1$	Itself	$p^4(p-1)$	$\mathscr{H}$	
	$\mathbf{G}_1(\mathbf{C})$	$p^3(p-1)$	$+\mathbf{C}$	
	$\mathbf{G}_1(\mathbf{H}^\infty)$	$p^2(p-1)$	$+\ \mathbf{H}^\infty \Longrightarrow (\mathscr{I}_U)$	
	$\mathbf{A}$	$p(p-1)$		$< \mathbf{G}_1(\mathbf{H}^\infty), < \mathbf{G}_1(\mathbf{C})$
	$\mathbf{G}_1(\mathbf{H}_C)$	$p(p-1)$	$+\mathbf{H}_C$	
	$\mathbf{T}_p$	p		Isomorphic to $\mathbb{Z}_P$, Translations along $\mathscr{H}$
	$\mathbf{GA}$	$p-1$		Isomorphic to $\mathbb{Z}_P^*$, Subgroup of $\mathbf{A}$
	$\mathbf{SA}$	p		Isomorphic to $\mathbb{Z}_P$, Determinant 1
$\mathbf{G}_2$	Itself		$\mathscr{H}$	
	$\mathbf{G}_2(P^\infty)$	$p^3(p-1)$	$+P^\infty$	
	$\mathbf{G}_2(\mathbf{C})$	$p^2(p-1)$	$+\mathbf{C}$	
	$\mathbf{G}_2(\mathbf{H}_C)$	$p(p-1)$	$+\mathbf{H}_C$	
	$\mathbf{G}_2(\mathbf{H}^\infty)$	$p(p-1)$	$+\mathbf{H}^\infty + P^\infty$	$< \mathbf{G}_1(\mathbf{H}^\infty), \in UT(2, F_p)$
	$\mathbf{G}^{\mathrm{sym}}$	$p-1$		Isomorphic to $\mathbb{Z}_P^*$
	$\mathbf{G}^{\mathrm{Imp}}$	p		Isomorphic to $\mathbb{Z}_P$, $\in \mathbf{I}_n, \mid I_n^2 = \mathbf{1}, \forall I_n \in \mathbf{I}_n$
$\mathbf{G}_{e_1}$	Itself		$\mathscr{I}_{\mathrm{U}}$	Exchange symmetry
$\mathbf{G}_{e_2}$	Itself		Same as above	In general can't be reduced from $\mathbf{G}_{e_1}$

Table 4: Table of all important transformations and symmetries for the hyperplane defined in equation (5.29). Of particular interest are invariances with the centre, the line at infinity, and the point at infinity. The exchange symmetry is also noted as a co-set of transformations. Subgroups and important properties are noted along with the relevant groups. For the particular parametrization refer to our results in tables 1,2,3.

which preserve structure. In this section a more general outlook on the enumeration of the groups shall be presented. For comparison we shall also compare the groups defined in (pg. 419-421 of [Hir79]). The groups are defined over the vector spaces and with their projective equivalents, such that either the vector space or the projective space is kept invariant. In this section, using the formulas for a few particular groups, I have tried to calculate the exact number of elements. The motivation for such an enumeration is to check for groups which might be isomorphic to the groups we have discussed above.

We first enumerate the groups of semi-linear transformations in the vector space $V(n,p)$ for 2 dimensions and the projective groups (given by the label P before the form) which are defined over $PG(n-1,p)$. The labels have the following meanings (taken from pg.419 of [Hir79]).

- I: All transformations over the vector space
- S: Transformations with determinant 1.
- G: Transformations in I up to a scalar factor.

The groups themselves are defined by the structures which are kept invariant in the projective space. We have:

- L: Semi linear Transformations keeping the entire space $PG(n-1,p)$ invariant. These are the projectivities.
- O: Orthogonal transformations which keep the quadratic form $x_1^2 + x_2x_3 + \ldots + x_{n-1}x_n$ invariant.
- U: Unitary transformations defined over the the field extension $q = p^h$, where h is a square. These keep invariant the form $x_1^{\sqrt{q}+1} \ldots x_n^{\sqrt{q}+1}$.
- Sp: Symplectic transformations which keep the null polarity invariant.

The forms corresponding to the above in the projective space have the label P attached to them.

An interesting transformation is the set of unitary transformations U, which is defined over a field extension of p. Here we look at only those transformations which are defined for the field extension p^2.

In the case of dimension 3, we have the transformations over the vector space $V(3,p)$ and over the projective space $PG(2,p)$, which we have so far worked with. In 3 dimensions however, there is no symplectic group, but the orthogonal group exists (since now we have the subspaces where the Euclidian quadric might be defined).

For the sake of completion the enumeration of groups in 4 dimensions are also given below.

Form	L	U	Sp
I	$p(p-1)(p^2-1)$	$p(p+1)(p^2-1)$	$p(p^2-1)$
S	$p(p^2-1)$	$p(p^2-1)$	$p(p^2-1)$
G	$p(p-1)(p^2-1)$	$p(p^2-1)^2$	$p(p^2-1)(p-1)$
P	$p(p^2-1)$	$p(p^2-1)$	$p(p^2-1)/2$
PS	$p(p^2-1)/2$	$p(p^2-1)/2$	$p(p^2-1)/2$
PG	$p(p^2-1)$	$p(p^2-1)$	$p(p^2-1)$

Table 5: Order of groups in 2 dimensions. Orders were calculated using the table on pages 420-421 in [Hir79].

Form	L	U	O
I	$p^3(p-1)(p^2-1)(p^3-1)$	$p^3(p+1)(p^2-1)(p^3+1)$	$2p(p^2-1)$
S	$p^3(p^2-1)(p^3-1)$	$p^3(p^2-1)(p^3+1)$	$p(p^2-1)$
G	$p^3(p-1)(p^2-1)(p^3-1)$	$p^3(p^2-1)^2(p^3+1)$	$p(p^2-1)(p-1)$
P	$p^3(p^2-1)(p^3-1)$	$p^3(p^2-1)(p^3+1)$	$p(p^2-1)$
PS	$p^3(p^2-1)(p^3-1)/2$	$p^3(p^2-1)(p^3+1)/2$	$p(p^2-1)$
PG	$p^3(p^2-1)(p^3-1)$	$p^3(p^2-1)(p^3+1)$	$p(p^2-1)$

Table 6: Order of groups in 3 dimensions. Calculated using the table on pages 420-421 in [Hir79].

Form	L	U	Sp
I	$p^6(p-1)(p^2-1)(p^3-1)(p^4-1)$	$p^6(p+1)(p^2-1)(p^3+1)(p^4-1)$	$p^4(p^2-1)(p^4-1)$
S	$p^6(p^2-1)(p^3-1)(p^4-1)$	$p^6(p^2-1)(p^3+1)(p^4-1)$	$p^4(p^2-1)(p^4-1)$
G	$p^6(p-1)(p^2-1)(p^3-1)(p^4-1)$	$p^6(p^2-1)^2(p^3+1)(p^4-1)$	$p^4(p-1)(p^2-1)(p^4-1)$
P	$p^6(p^2-1)(p^3-1)(p^4-1)$	$p^6(p^2-1)(p^3+1)(p^4-1)$	$p^4(p^2-1)(p^4-1)/2$
PS	$p^6(p^2-1)(p^3-1)(p^4-1)/2$	$p^6(p^2-1)(p^3+1)(p^4-1)/2$	$p^4(p^2-1)(p^4-1)/2$
PG	$p^6(p-1)(p^2-1)(p^3-1)(p^4-1)$	$p^6(p^2-1)(p^3+1)(p^4-1)$	$p^4(p^2-1)(p^4-1)$

Table 7: Order of groups in 4 dimensions. Calculated using the table on pages 420-421 in [Hir79].

Isomorphisms

Since the enumeration of the groups has been done, one can now ask whether there exists an isomorphism between the groups we have probed and the groups known to exist. Here one may note that for the case of the 2-dimensional Unitary, the 2-dimensional symplectic, and the 3-dimensional orthogonal groups, one has an extra factor of (p^2-1) which is missing from the enumerations of the groups we have dealt with so far. For such an isomorphism to take place the existence of this factor is a must, otherwise a one-to-one correspondence will not be established.

This extra factor is in fact a part of our calculations, and was mentioned before. These are in fact the (p^2-1) translations which we need to take into account. The groups have so far been discussed for particular translations, however, such a choice of a particular hyperplane of intersection seems arbitrary. The degree of freedom of chosing these transformations therefore allows the (p^2-1) factor to come into play.

We notice in particular the isomorphism between the groups $SU(2d)$ and $PGU(2d)$ with the groups **SA** and $\mathbf{T}_p$. The other isomorphism exists between the groups **A**, keeping the line at infinity and the centre invariant with the group $U(2d)$.

The main motivation for the form of gauge transformations comes from the isomorphism

$$\Phi : \mathbf{SA} \to \mathrm{O}(3d) \tag{6.35}$$

where $\mathrm{O}(3d)$ is the orthogonal group acting on projective space of dimension 2. This leads us to finally look at these orthogonal transformations and why they are important.

6.3 Automorphisms of the 2-dimensional quadric

As we have seen before the automorphisms of the 2-dimensional quadric has the subgroup of elements which are the Lorentz transformation. However, there exists another case, such as was seen in the the case of 1-dimensional quadric in section 5. These are those transformations which don't keep the centre invariant, and are of the form:

$$O = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0}^T & \Lambda \end{pmatrix} \tag{6.36}$$

Due to the condition that $O^{-T}Q^{+}O^{-1} = Q^{+}$, we have that the matrix representative $\Lambda \subset \mathbf{O}(2)$, where we have $\mathbf{O}(2)$ the set of orthogonal matrices in 2-dimensions with p elements.

It becomes clear what exactly O represents when we look slightly deeper. We notice first that the form is the same as those for rotations in 2 spatial dimensions. Therefore we have that in the 3-dimensional vector space, O corresponds to rotations about the 'time' axis (given by the hyperplane $\{0,1,0\}$), such that the hyperplane which is the plane $A = \{1,0,0\}$ is kept invariant since $O^{-T}A = A$. The action for the form with determinant 1, which are the proper transformations with $(p-1)/2$ elements[31], given by,

$$\Lambda = \begin{pmatrix} \lambda & \pm\sqrt{1-\lambda^2} \\ \mp\sqrt{1-\lambda^2} & \lambda \end{pmatrix} \tag{6.37}$$

can be seen on the centre: $\{0,0,1\}^T \rightarrow \{0, \pm\sqrt{1-\lambda^2}, \lambda\}^T \rightarrow \{0, \pm\sqrt{1-\lambda^2}/\lambda, 1\}^T$, such that the centre is translated along the spatial axis in the affine plane. A more in-depth discussion about these transformations in 2 and 4 dimensions can be found in section 3.2 of [Pes19].

That these transformations are important in the consideration of gauge transformations shall become clear soon. First we notice in figure 16, the existence of the local world domain with two quadric points in the affine plane.

The fiber space which was defined back in definition 3.1, is constructed by connecting quadric points inside the local world domain to those outside it. This idea is chosen since all quadric points, irrespective of their locations in the two domains are at a distance of 1 from the center. The choice of this fiber space has two particular degrees of freedom:

1. The choice of the affine plane chosen and therefore the choice of which quadric points to put in the local world domain. In particular, we have set the affine plane with the last coordinate as 1. But this is arbitrary, and any other choice would suffice as well. The local world domain in other 'affine planes' might have different quadric points and not the two we have seen above.

2. The choice of the centre itself is arbitrary, and has only been chosen for convenience of calculation. That any other point can be chosen is clear, as is the fact that the local world domain shall also shift as the centre does[32].

The first point relates directly to the second subgroup of the automorphism group[33], namely the orthogonal group defined in equation (6.37). These

[31] This is the normal subgroup of the group of orthogonal matrices.
[32] This is because the distance is defined with respect to the chosen centre.
[33] The first are of course the Lorentz transformations.

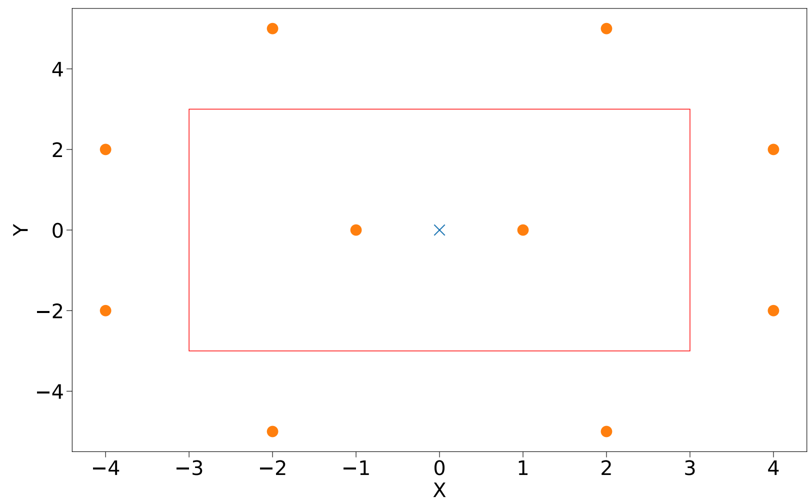


Figure 16: Minowski quadric for prime $p = 11$ shown in orange. The square shows the local world domain such that all points inside it will be mapped back to themselves if the square root of their squared values are taken. The edge points of the local world domain are at a distance of 0 from the centre, and represent points on the light cone quadric. Out of a total of 10 affine quadric points only 2 are in the local world domain.

are precisely those transformations which represent the quadric point based symmetries of the fiber space, and the discussion in the 1-dimensional case as given in equation (5.8) sets up this idea for higher dimensions. These transformations map quadric points to other quadric points in such a way that the local world domain can be defined freely for any chosen set of quadric points.

The second point makes it clear that one must consider other points as centres, and therefore we consider translations of the centre. The hyperplane of intersection is the second consideration in the symmetry of the gauge transformations. It represents precisely that part of the local world domain which is the intersection of the local world domains of any two centres.

Therefore, instead of looking at transformations which keep A invariant, one can ask for transformations keeping either the hyperplane of intersection invariant or the hyperplane perpendicular to it. First, we call the point set P_q as the set of points which are in the quadric Q^+. Then the transformations O map the points in this point set to each other such that the quadric and the corresponding form is kept invariant.

We now choose a projectivity ρ, such that we have $P'_q = \rho P_q$, where P' is the set of new points, and $\rho^{-T}A = A_{\mathscr{H}}$ or $\rho^{-T}A = \mathscr{H}$. The axis $A_{\mathscr{H}}$ is given by:

$$A_{\mathscr{H}} = \begin{pmatrix} t_2 \\ t_1 \\ 0 \end{pmatrix} \tag{6.38}$$

such that in the 3-dimensional vector space, $A_{\mathscr{H}}$ is perpendicular to $\mathscr{H}$.

It is clear that there are only two types of transformations which have $P'_q = P_q$ such that the quadric is invariant. These are either Lorentz transformations or the transformations belonging to O. However, since O keeps A invariant, one can not use these transformations.

Before the Lorentz case however, a general projectivity can be used. This leads to a new quadric form $\rho^{-T}Q^{+}\rho^{-1}$, which represents the point set P'_q.

Theorem 6.2. *Given a general projectivity ρ, we have the transformation of O as:*

$$O' = \rho O \rho^{-1} \tag{6.39}$$

where O' keeps the new quadric $Q' = \rho^{-T}Q^{+}\rho^{-1}$ invariant.

Proof. *For O', we have:*

$$O'^{-T}Q'O'^{-1} = (\rho O \rho^{-}1)^{-T}Q'(\rho O \rho^{-}1)^{-1} \tag{6.40}$$

$$= \rho^{-T}O^{-T}\rho^{T}Q'\rho O^{-1}\rho^{-1}$$

$$= \rho^{-T}O^{-T}QO^{-1}\rho^{-1}$$

$$= \rho^{-T}Q\rho^{-1} = Q'$$

where we have used that $O^{-T}QO^{-1} = Q$.

□

Remark. *O' keeps the new hyperplane $H = \rho^{-T}A$ invariant.*

For the projectivity ρ we have the two forms given by:

$$\rho^{-T} = \begin{pmatrix} A_{\mathscr{H}} & A' & H^{\infty} \end{pmatrix} \tag{6.41}$$

$$\rho^{-T} = \begin{pmatrix} \mathscr{H} & A' & H^{\infty} \end{pmatrix} \tag{6.42}$$

where A' defines the freedom in choosing different projectivities, and is the image of the hyperplane $\{0, 1, 0\}$.

The special projectivity given by:

$$\rho^{-T} = \begin{pmatrix} \mathscr{H} & A_{\mathscr{H}} & H^{\infty} \end{pmatrix} \tag{6.43}$$

maps the plane $\{1,0,0\}^T$ to the hyperplane of intersection $\mathscr{H}$ and the time axis $\{0,1,0\}$ to the hyperplane $A_{\mathscr{H}}$. The exact form of the transformation O' is quite cumbersome to write, and we first limit ourselves to the case we have already talked about before, with the translation vector $\mathbf{t} = \{1,0,0\}^T$.

Definition 6.2. *Transformations for the generator*

We have for $\boldsymbol{t} = \{1,0,0\}$*:*

$$\mathscr{H} = \{-2,0,1\}^T, \quad A_{\mathscr{H}} = \{0,1,0\}^T$$

such that for the projectivity defined by:

$$\rho^{-T} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

we have the new transformation

$$O' = \begin{pmatrix} 1 & \mp\gamma/2 & (\lambda-1)/2 \\ 0 & \lambda & \pm\gamma \\ 0 & \mp\gamma & \lambda \end{pmatrix}, \quad \gamma = \sqrt{1-\lambda^2}, \tag{6.44}$$

with the quadric represented by:

$$Q' = \begin{pmatrix} -4 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix} \tag{6.45}$$

The transformation (6.44) keeps the point at infinity invariant, but not the line at infinity or the hyperplane of centres. Therefore this is a special subgroup of the group $\mathbf{G}_2(\mathbf{P}^\infty)$.

For the case of the generator the quadric points and the transformed quadric points are plotted in figure 17. We note again that the orthogonal transformation keeps the initial quadric invariant along with the hyperplane $\{1,0,0\}^T$, while the new quadric is kept invariant by the transformed elements, along with the hyperplane of intersection for the generator. Interesting to notice is the inclusion of the centre in the new quadric.

Other forms of projectivities are of the form $\rho^{-T}A = A_{\mathscr{H}}$, and we notice first and foremost that for the translation with $\mathbf{t} = \{0,1,0\}$, we have the same hyperplane, and the identity as the projectivity. For the projectivity given by the form:

$$\rho^{-T} = \begin{pmatrix} t_2 & t_1 & 0 \\ t_1 & t_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.46}$$

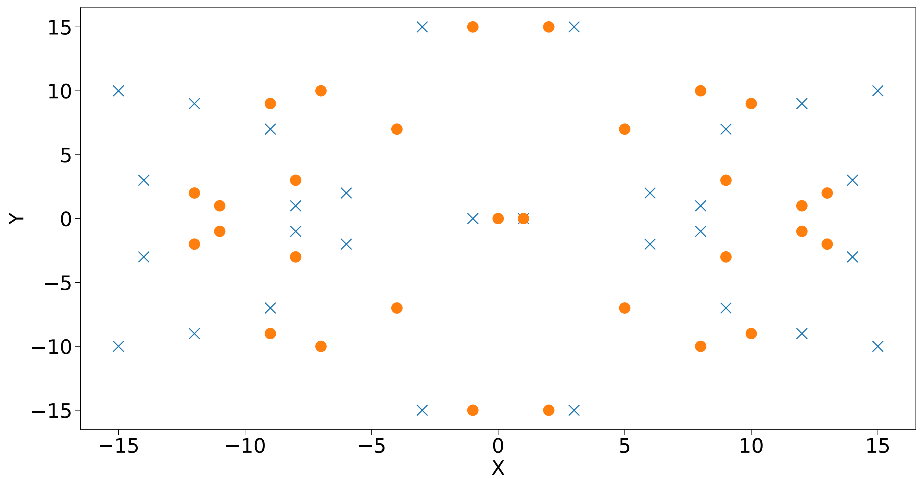


Figure 17: Blue crosses represent the original quadric points. The new quadric points (transformed according to equation (6.45)) are given by the orange dots, with one point belonging to both the quadrics. The points at infinity are omitted here for the sake of clarity.

there exist certain elements namely when $t_1^2 = t_2^2 - 1$, or that $t_m = 1$, such that the projectivities are the Lorentz transformations. In this case the new quadric form remains the same as the old one.

7 Summary and Outlook

Once upon a midnight dreary, while I pondered, weak and weary, over many a quaint and curious volume of forgotten lore— While I nodded, nearly napping, suddenly there came a tapping, as of some one gently rapping, rapping at my chamber door. "'Tis some visitor," I muttered, "tapping at my chamber door— Only this and nothing more."

Edgar Allan Poe, The Raven

In this thesis the symmetries of the finite projective space were investigated. Following previous work, the Lorentz group as the automorphism group of the biquadric was represented and a visual investigation revealed how the finite group acts. In particular the action of the boosts and rotations were seen on the two-dimensional subspaces. The axial and point-symmetric nature of these transformations was confirmed. The graphical representation also seperated the Local World Domain from other points of the space when a second order distance was introduced.

The primary idea of connecting the local domain with the points lying outside was the introduction of a fiber space which connects quadric points inside and outside this local domain. In this thesis this idea was further developed to look at the intersection of two biquadrics. The intersection was found to be a hyperplane which was called the hyperplane of intersection. The symmetry groups of this hyperplane of intersection were parametrized, first for a canonical translation and then for a general one. Of particular note were the symmetries along with the centre, the direction of translation, and the line at infinity. The line of infinity when added to the hyperplane of intersection resulted in a bigger structure known as the interunion. The symmetries of this interunion were also found, where the normal subgroup kept the hyperplane of intersection and the line at infinity simultaneously invariant, and its co-set the exchange transformations inverted the two structures.

Higher order parametrizations were not explicitly noted, but a generalized approach was outlined. In the end the order of groups of the projective space, including linear, orthogonal, symplectic, and unitary groups were found. These orders allow the existence of isomorphisms with previous transformations of the hyperplanes. Finally, the two-dimensional automorphism group of the quadric was introduced, which led to the orthogonal transformations. The general form of this group was found, and the transformed representation was calculated for when the hyperplane of intersection is also kept invariant.

This thesis has answered questions about forms of symmetry groups, their orders, and their subgroups, especially for the 2-dimensional case. The gauge group and it's parameterization for 2-dimensions has been found as well. However, many questions have also emerged, and here they are enumerated:

1. What is the exact form of gauge transformations for higher dimensions, in particular in 3 and 4 dimensions?

2. What role does the hyperplane of intersection play for the construction of the fiber space?

3. What is the exact structure of the fiber space?

4. Given a fiber space and it's gauge symmetries how exactly does one interpret them as particles? If these particles are fluctuations in a homogenous biquadric field, how may they be quantized?

5. How does the structure of the hyperplane of intersection and the gauge transformations change when one approaches the continuum limit? Is the existence of a continuum limit perhaps the motivation behind the existence of higher dimensional gauge transformations?

In section 5.4 we gave an outline about how symmetry groups might be parameterized in the case of 3 dimensions. Here we consider a particular orthogonal group in 3 dimensional projective space given by

$$\mathbf{O}_{3d} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0}^T & \mathbf{O} \end{pmatrix}$$

with the property that $\mathbf{O}^T\mathbf{O} = \mathbf{I}$. This group is the 3-dimensional 'extension' of the gauge group given in equation (6.36) for 2 dimensions, where we now represent Λ as $\mathbf{O}_{2d}$ for the sake of clarity. Furthermore $\mathbf{O}_{3d}$ is again a subgroup of $\mathrm{Aut}(Q_{3d})$, which are the automorphisms of the quadric in 3-dimensions. The representation matrix $\mathbf{O}$ can further be written down into two forms represented as

$$\mathbf{O}_S = \begin{pmatrix} \mathbf{O}_{2d} & \mathbf{0}^T \\ \mathbf{0} & 1 \end{pmatrix}, \quad \mathbf{O}_R = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0}^T & \mathbf{O}_{2d} \end{pmatrix}$$

We note here now that $\mathbf{O}_S$ has the same form as the subgroup of spatial rotations given in equation (4.11). Similarly $\mathbf{O}_R$ has the same form as our gauge group in 2-dimensions. Therefore the intersection of the gauge group and the Lorentz group in 3-dimensions is not merely the identity element. This 'result' underlies a shift between symmetry groups in 2 dimensions to those in 3 dimensions. Understanding this distinction requires further research outside of the scope of this master's thesis. However the work done here could be used as a starting point to support the notion of gauge transformations in higher dimensions.

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A Supplementary equations and discussions

A.1 The affine part of the Hyper-plane

The group of transformations that keep the interunion invariant is a set which contains the group of transformations that keep the hyper-plane invariant. In the affine plane, with $p_3 = 0$, one gets a line in the affine plane, with the reduced representation which can be represented as:

$$\mathscr{H}_{\text{aff}} = \begin{pmatrix} \frac{2t_1}{t_m} \\ \frac{-2t_2}{t_m} \\ 1 \end{pmatrix}, \quad t_m = \boldsymbol{t}^T G_{\min} \boldsymbol{t}$$

The group of transformations which keep this line invariant are given by as:

$$\mathscr{H}_{\text{aff}} = \mathbf{G}_{\text{aff}}^T \mathscr{H}_{\text{aff}}$$

where $\mathbf{G}_{\text{aff}}$ can be solved using the affine representation above to give:

$$\mathbf{G}_{\text{aff}} = \begin{pmatrix} g_1 & \frac{-t_2(1-g_2)}{t_1} & 0 \\ \frac{-t_1(1-g_1)}{t_2} & g_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad g_1, g_2 \in F_p$$

These transformations however, keep the intersection invariant by choosing from points in the space-time which lie on the same hyperplane. It is possible that these other points might not be quadric points. The effect of the transformation on the complete quadric does not lead to a new quadric. However, one can find parameterized transformations which only work on the intersection of the two quadrics, and not the complete hyperplane of intersection. Such a transformation has been talked about before.

A.2 General automorphisms of the quadric

For transformations which keep the Minkowski quadric invariant but not the centre, we have:

$$\mathbf{G} = \begin{pmatrix} \Lambda_{2\times 2} & \alpha_{2\times 1} \\ \beta_{1\times 2} & 1 \end{pmatrix}$$

To give:

$$\Lambda^T I_2 \Lambda + \beta^T \beta = I_2$$
$$\Lambda^T I_2 \alpha + \beta^T = 0$$

$$\alpha^T I_2 \alpha = 0,$$

where we have written the top 2×2 part of the Minkowski metric as I_2.

Now, one can ask, for what solutions of the above do we also keep the centre invariant? And of course the answer is the Lorentz transformations which we have discussed before. A more specific question then would be to ask, if there exists any transformation, apart from the identity, which keeps not just the initial Minkowski quadric but also the translated quadric invariant? The answer is yes. There is a transformation, which one gets when one also solves the equations for the second quadric, which is the same as the transformation defining the Quadric Symmetry Group, given as:

$$\mathfrak{L} = \begin{pmatrix} -\frac{t_e}{t_m} & \frac{2t_1t_2}{t_m} & 0 \\ -\frac{2t_1t_2}{t_m} & \frac{t_e}{t_m} & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad t_e = t_1^2 + t_2^2$$

A.3 Action of the exchange symmetries of type-1 and a note on parameterization

Group G_{e_1} is one of the sets of transformations which keep the interunion invariant. An explicit representation is given by:

$$\mathbf{G}_{e_1} = \begin{pmatrix} g_1 & \frac{t_2(g_2+1)}{t_1} & g_3t_2 \\ \frac{t_1(g_1+1)}{t_2} & g_2 & g_3t_1 \\ \frac{2t_1}{t_m} & -\frac{2t_2}{t_m} & 1 \end{pmatrix}; \quad g_1, g_2, g_3 \in F_p$$

With such a parametrization for every g_1, g_2, g_3 we have $p^2-(2p-1)-2(p-1)$ matrices. This is because neither of the t's can be 0, nor can t_m be 0. We then have p^3 choices for the parameters. This gives us a total of $p^3(p-3)(p-1)$ matrices. However, for the case of only invertible matrices, one has $p^3 - p^2$ choices, since $|\mathbf{G}_{e_1}| = -(g_1+g_2+1)$, and so in total there are $p(p+1)(p-1)^2(p-3)$ non-singular transformations. This is clear to check for example for the case of $p = 3$ where there are indeed 0 such transformations. One can see that for keeping the hyper-plane of centres invariant, the above group only has one element, where $g_1 = -g_2 = \frac{(t_1^2+t_2^2)}{t_m}; g_3 = 0$. For the case when only the initial centre is kept invariant one has $g_3 = 0$. For the case of keeping both the centres invariant, one employs the use of homogenous coordinates (since the last coordinate is -1 for second centre), to get:

$$\mathbf{G}^C_{e_1} = \begin{pmatrix} g_1 & -\frac{t_1(g_1+1)}{t_2} & 0 \\ \frac{t_1(g_1+1)}{t_2} & -(1+\frac{t_1^2(g_1+1)}{t_2^2}) & 0 \\ \frac{2t_1}{t_m} & -\frac{2t_2}{t_m} & 1 \end{pmatrix}; g_1 \in F_p$$

An order can also be found for a such a parametrization for $\mathbf{G}_{e_2}$, counting only non-singular matrices. These are total of $p^3 - p = p(p-1)(p+1)$ elements, equaling the order also for $\mathbf{G}_{e_1}$.

However, for $\mathbf{G}_1$ a similar parametrization will have the change that neither of t_1, t_2 is 0, and therefore the order of $\mathbf{G}_1$ will be $p(p-1)^3(p+1)$, which is greater than $\mathbf{G}_{e_1}$. Since we have already seen that the two orders should in fact be equal, one notes that such a straight crude parametrization might lead to a lack of degrees of freedom in the general case and therefore must be avoided.

A.4 Elements of groups

Another interesting point is counting matrix elements for groups for p=3, which have three matrices (two are inverse, and then one is the identity). So for keeping the hyper-plane, the line connecting the two centres, and the two centres (point-wise) invariant one has:

$$\mathbf{G}_1(\mathrm{H_C},\mathrm{C}) = \begin{pmatrix} 0 & \frac{t_1}{t_2} & 0 \\ \frac{-t_2}{t_1} & 2 & 0 \\ \frac{-2}{t_1} & \frac{2}{t_2} & 1 \end{pmatrix}, \quad \mathbf{G}_2(\mathrm{H_C},\mathrm{C}) = \begin{pmatrix} 2 & \frac{-t_1}{t_2} & 0 \\ \frac{t_2}{t_1} & 0 & 0 \\ \frac{2}{t_1} & \frac{-2}{t_2} & 1 \end{pmatrix}, \quad \mathbf{I}$$

Similarly, there exist 9=8 + 1 transformations for keeping the hyperplane of intersection and only the two centres point wise invariant:

$$\begin{pmatrix} 0 & \frac{t_1}{t_2} & 0 \\ \frac{t_2}{t_1} & 0 & 0 \\ \frac{2t_e}{t_1 t_m} & \frac{-2t_e}{t_2 t_m} & 1 \end{pmatrix}, \begin{pmatrix} 0 & \frac{t_1}{t_2} & 0 \\ 0 & 1 & 0 \\ \frac{2t_1}{t_m} & \frac{2t_1^2}{t_2 t_m} & 1 \end{pmatrix}, \begin{pmatrix} 0 & \frac{t_1}{t_2} & 0 \\ \frac{-t_2}{t_1} & 2 & 0 \\ \frac{-2}{t_1} & \frac{2}{t_2} & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{t_2}{t_1} & 0 & 0 \\ \frac{2t_2^2}{t_m t_1} & \frac{-2t_2}{t_m} & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ \frac{-t_2}{t_1} & 2 & 0 \\ \frac{-2t_2^2}{t_m t_1} & \frac{2t_2}{t_m} & 1 \end{pmatrix}, \mathbf{I}$$

$$\begin{pmatrix} 2 & \frac{-t_1}{t_2} & 0 \\ \frac{t_2}{t_1} & 0 & 0 \\ \frac{2}{t_1} & \frac{-2}{t_2} & 1 \end{pmatrix}, \begin{pmatrix} 2 & \frac{-t_1}{t_2} & 0 \\ 0 & 1 & 0 \\ \frac{-2t_1}{t_m} & \frac{2t_1^2}{t_2 t_m} & 1 \end{pmatrix}, \begin{pmatrix} 2 & \frac{-t_1}{t_2} & 0 \\ \frac{-t_2}{t_1} & 2 & 0 \\ \frac{-2t_e}{t_m t_1} & \frac{2t_e}{t_2 t_m} & 1 \end{pmatrix}$$

And similarly, there are 9= 8 + 1 transformations for keeping the line connecting the two centres and the hyperplane of intersection invariant.

$$\begin{pmatrix} 0 & \frac{-t_1}{t_2} & 0 \\ \frac{-t_2}{t_1} & 0 & 0 \\ \frac{-2}{t_1} & \frac{-2}{t_2} & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ \frac{-t_2}{t_1} & 1 & 0 \\ \frac{-2}{t_1} & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \frac{t_1}{t_2} & 0 \\ \frac{-t_2}{t_1} & 2 & 0 \\ \frac{-2}{t_1} & \frac{2}{t_2} & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & \frac{-t_1}{t_2} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{-2}{t_2} & 1 \end{pmatrix}, \begin{pmatrix} 1 & \frac{t_1}{t_2} & 0 \\ 0 & 2 & 0 \\ 0 & \frac{2}{t_2} & 1 \end{pmatrix}, \mathbf{I}$$

$$\begin{pmatrix} 2 & \frac{-t_1}{t_2} & 0 \\ \frac{t_2}{t_1} & 0 & 0 \\ \frac{2}{t_1} & \frac{-2}{t_2} & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ \frac{t_2}{t_1} & 1 & 0 \\ \frac{2}{t_1} & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & \frac{t_1}{t_2} & 0 \\ \frac{t_2}{t_1} & 2 & 0 \\ \frac{2}{t_1} & \frac{2}{t_2} & 1 \end{pmatrix}$$

One can see the two transformations (1 transformation and it's inverse) for the case in the hyper-plane of centres, where the second centre is mapped on to itself.

A.5 Parametrisation of $P_{\mathbf{exch}}$

For P_{exch} one writes again, $P^T P^T l^\infty = l^\infty$, and so,

$$P_{\text{exch}} = \begin{pmatrix} \lambda & \alpha \\ \beta & 1 \end{pmatrix} \implies \lambda^T \beta^T + \beta^T = \mathbf{0}_{2\times 1}, \alpha^T \beta^T = 0$$

Note: For groups G_1 and G_2 the determinant is $(g_1 + g_2 - 1), -(g_1 + g_2 + 1)$ respectively.

A.6 Understanding the hyperplane of intersection

In the 2-d case the smallest structure which contains two points is a line. Asking to keep 4 neighborhood points invariant imposes a condition which can be fulfilled only by the identity element. In such a case therefore we look at the lines which connect these two points. We end up with two lines, since it makes sense to have an affine part separate from the line at infinity. This effectively means that the group of transformations keeping the hyperplane of affine points invariant is the permutation group of $(p+1)$ points, whereas the interunion requires a permutation of $(2p+1)$ points. It is easy to see that line at infinity is the hyperplane dual to the centre, meaning it is defined by the centre and the biquadric centres on it. A translation of the centre to another point also transforms the biquadric centred at the new centre, according to the rules of projectivities. However, this transformation preserves

the hyper-plane at infinity, and therefore this hyper-plane is in fact dual to all the possible centres with respect to the translated quadrics on them. This is also why translation doesn't change the neighborhood at infinity.
Similarly one can ask, are there any properties of duality associated with the hyperplane $\mathscr{H}$? One sees immediately from (5.14) that in fact this is the hyperplane dual to the point $P_d = \{\frac{-2t_1}{t_m}, \frac{-2t_2}{t_m}, 1\}$ with respect to the initial Minkowski quadric. This point is actually a part of the hyperplane of centers. A similar point exists in the future, which is this point reflected through the center $\{\frac{2t_1}{t_m}, \frac{2t_2}{t_m}, 1\}$. The hyperplane of intersection then is also the dual hyperplane with respect to the new translated quadric and the translation of this new point. One writes this as:

$$\mathscr{H} = Q_{\text{min}} P_d$$

$$\text{and } \mathscr{H} = Q_{\text{trans}} T R P_d$$

$$R = \text{diag}(-1,-1,1), \quad T = \text{Translation by } \mathbf{t}$$

The above discussion motivates us to consider instead a family of hyperplanes. Given an initial centre with a biquadric centred on it, and a direction of motion (given by hyperplane of centres), the entire family of future and past centres (i.e. every point of the line of centres) gives rise to a family of dual hyperplanes in the affine plane with respect to the initial quadric. This family of hyperplanes is the set of all the possible hyperplanes of intersection. It is clear then, that any transformation which keeps the initial quadric invariant along with the hyperplane of centres, must then also keep this family of hyperplanes of intersection invariant.

The last result follows from the fact that a transformation which keep a point and a quadric invariant must keep the hyperplane dual to it also invariant (with respect to the quadric). This is easy to see for instance with Lorentz transformations; since they keep the Minkowski quadric and the center invariant, they also keep the line at infinity invariant.

A.7 Case of purely vertical translations ($t_1 = 0$)

For this case we still have the same points at infinity, but the points in the affine plane are given by: $\mathbf{P} = \{\pm\sqrt{t_2^2+1}, t_2/2, 1\}$, defining a hyper-plane $\mathscr{H} = \{0, -2, t_2\}^T$. The interunion is the trivial form with t_m replaced by t_2 after factoring out a t_2. The transformations keeping the hyperplane and interunion invariant are then given by:

$$\mathbf{G}_{\mathscr{H}} = \begin{pmatrix} g_1 & g_2 & g_3 \\ \frac{t_2 g_4}{2} & g_5 & 0 \\ g_4 & \frac{2(g_5-1)}{t_2} & 1 \end{pmatrix}, \quad \mathbf{G}_{\mathrm{IU}} = \begin{pmatrix} g_1 & g_2 & g_3 \\ 0 & g_4 & 0 \\ 0 & \frac{1-g_4}{t_2} & 1 \end{pmatrix}; \quad g_4 = \pm 1 \tag{3.26}$$

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[34]Defined as the sense of self I have at this point in spacetime.

Statutory Declaration

I confirm here that I have written the accompanying master's thesis by myself, without contributions from any sources other than those cited in the text. This thesis was not submitted to any other authority to achieve an academic grading and was not published elsewhere. Furthermore I declare that the submitted written copy of the present thesis and the version submitted electronically are the same.

Place, Date

Name and Signature